

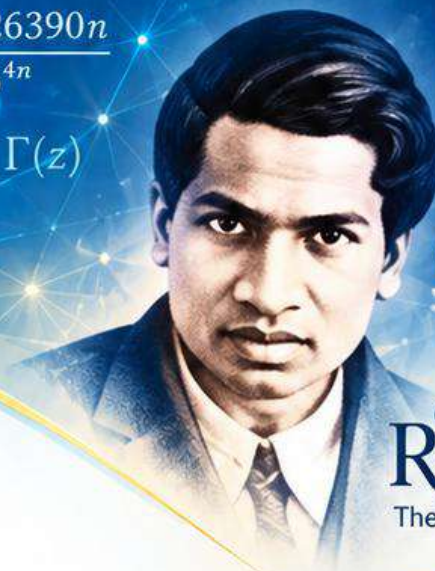
$$\sum_{n=0}^{\infty} \frac{1}{n!} = e^x \quad \frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}$$

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad \tau(n) \quad \Gamma(z)$$

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$q = e^{2\pi i \tau}$$

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$$



**Srinivasa
Ramanujan**

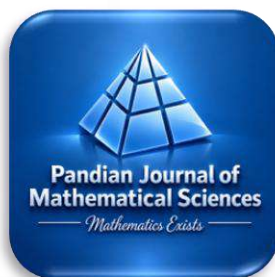
The Mathematical Genius
of Tamil Nadu

(1887 – 1920)

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$$e^{i\theta} = \cos \theta + i \sin \theta$$



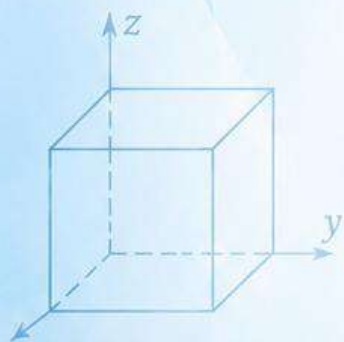
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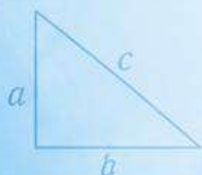
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$$a^2 + b^2 = c^2$$

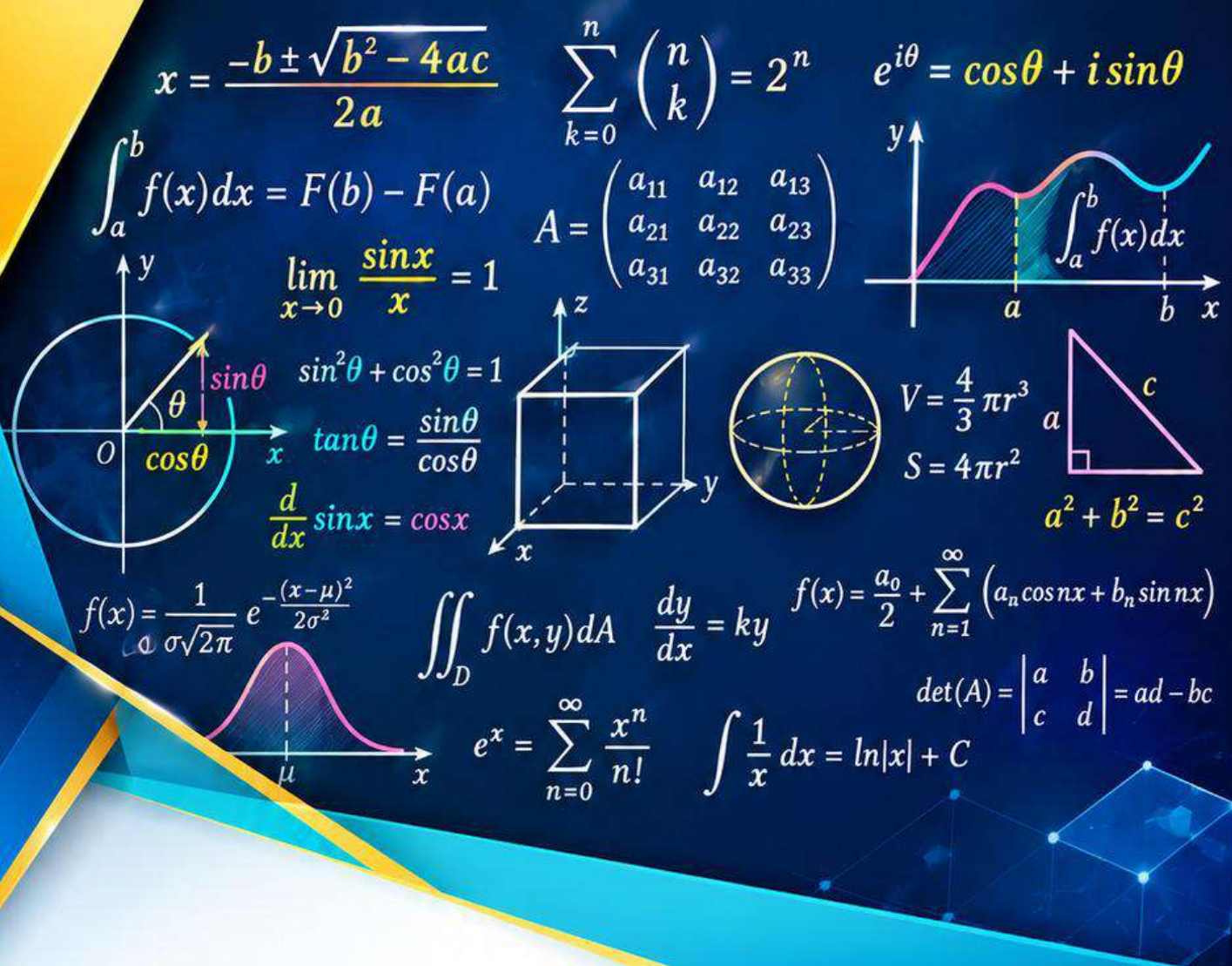


$$\int_a^b f(x) dx$$



$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$



Srinivasa Ramanujan – Mathematical Genius of Tamil Nadu

Srinivasa Ramanujan (1887–1920) was a brilliant Indian mathematician born in Erode, Tamil Nadu. Largely self-taught, he made extraordinary contributions to **number theory, infinite series, continued fractions, partitions, and mathematical analysis**.

His collaboration with **G. H. Hardy** at Cambridge brought international recognition to his work. His famous discoveries include the **Hardy–Ramanujan formula for partitions** and remarkable infinite series for π . He was elected a **Fellow of the Royal Society** in 1918.

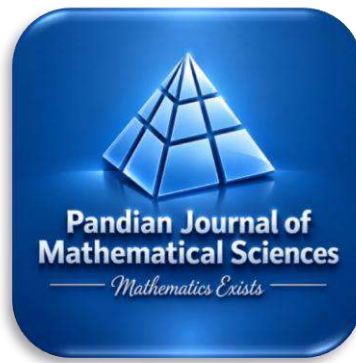
Though he died at only 32, his notebooks continue to inspire modern mathematical research. Ramanujan remains one of India's most celebrated mathematical geniuses.

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17 PARTNERSHIPS FOR THE GOALS



CONTENTS

S.No	Title / Author Name	Page No
1.	The Universal Pythagorean Angle Lattice: Converging to $\pi, e, \gamma, \zeta(3)$, Catalan's Constant, and the Golden Ratio φ Dr. Chetansing Rajput	1
2.	DJ Trigonometry A Square-Based Proportional Framework for Closed Geometric Figures Dheeraj Badwan	13
3.	Volumetric Sensible-Enthalpy Formulation for Coupled Transport in MHD Casson Nanofluid Flow: Revealing Energy Pathways beyond Wall Fluxes Felix Tega ONOJOVWO¹, Anselm Onyekachukwu OYEM²	30
4.	Iwuamadi's Primality Test Prof. Lawrence IWUAMADI	55
5.	The Impact of Waning Immunity on Covid-19 Dynamics: A Sqit Model with Contact Tracing and Quarantine Odebiyi, O.A^{1*}, Omoniyi, S.O¹.	61
6.	1. A PROOF OF COLLATZ'S CONJECTURE 2. A SHORT ESSAY ON OPPERMANN'S CONJECTURE AND ITS IMPACT ON GOLDBACH'S HYPOTHESIS 3. If $f(a) = 1 - a^p - (1 - a)^p \equiv 0 \pmod{p^3}$, then $f(a_r) \equiv 0 \pmod{p^{2r+1}}$ and 3^{p-1} is not $\equiv 1 \pmod{p^3}$, if $p \equiv 1 \pmod{6}$ Ramaswamy Krishnan	82
7.	Radial Schrödinger Equation on the Einstein Schwarzschild Manifold and FLRW Manifold Riadh Jelloul	91
8.	Radial Solution Of The Monge-Ampere Equation On The Complex Grassmannian $G_{2,4}\mathbb{C}$ Riadh Jelloul	99

9.	A Visual Exploration of the Inequality $1^e < e^1$: Enhancing Student Understanding Through Geometric Reasoning Dr. Md Sadikur Rahman	111
10.	Promoting Risk-Taking in Mathematics Classrooms through Emotionally Safe Learning Environments: A Narrative Review of Literature Dr. Santoshi Misra & Dr. D. Sirisha	116

The Universal Pythagorean Angle Lattice: Converging to π , e , γ , $\zeta(3)$, Catalan's Constant, and the Golden Ratio φ

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Abstract

We introduce the *Pythagorean angle lattice*, a countable set of real numbers obtained by applying the cotangent function to rational divisions of finite sums of acute angles of primitive Pythagorean triples. We prove that a distinguished single-partner-angle sublattice of this structure is *constructively universal*: for every positive real constant $T > 0$, and for any starting primitive Pythagorean triple with larger acute angle α_0 , there exists an explicit sequence of primitive Pythagorean triples such that

$$\cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{D}\right) \rightarrow T,$$

where D is an integer denominator determined by T and α_0 , and $\phi(u_k, v_k)$ is the Euclid angle of the partner triple. The sequence is generated by the continued fraction convergents (with a parity correction using semiconvergents) of an explicit target ratio $t = \tan((\text{Darccot}(T) - \alpha_0)/2)$. The convergence rate is $O(1/u_k^2)$ for subsequences of true convergents, and $O(1/u_k)$ for semiconvergents.

We demonstrate the universality of the construction with five pillar constants: π , Euler's number e , the Euler–Mascheroni constant γ , Apéry's constant $\zeta(3)$, and Catalan's constant G , and we establish the golden ratio φ as an exact special case. The framework unifies algebraic constants (metallic means and the golden ratio φ), transcendental constants (π, e), and constants whose irrationality remains open ($\gamma, \zeta(3), G$), within a single geometric construction arising from integer right triangles. The general multi-angle lattice is left for future investigation.

Keywords: Pythagorean triples, continued fractions, metallic means, golden ratio, universal approximation, π , e , Euler–Mascheroni constant, Apéry's constant, Catalan's constant, cotangent, primitive triples, Diophantine approximation.

1. Introduction

The metallic means, introduced by de Spinadel [1], are the positive roots of $x^2 - mx - 1 = 0$:

$$M_m = \frac{m + \sqrt{m^2 + 4}}{2}.$$

In recent work [2], a connection was established between these metallic means and primitive Pythagorean triples. For a primitive Pythagorean triple (a, b, c) with $c - b \in \{1, 2, 8\}$,

$$\cot\left(\frac{\theta}{4}\right) = M_n,$$

where $n = 2a/(c - b)$, θ is the smaller acute angle, and M_n is the n -th metallic mean. This identity is universally applicable for metallic means of order $n \geq 5$. The cases $n = 1, 2, 3, 4$ require separate treatment [2]. This connected the metallic means – an

algebraic family of constants – to the geometry of right triangles. A closely related cotangent normalization yields the Golden ratio

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

exactly, for *every* primitive Pythagorean triple (a, b, c) with $a < b < c$, with no restriction whatsoever, when composed with the fundamental $(3, 4, 5)$ triple via the Brahmagupta–Fibonacci identity: writing $\alpha_0 = \arctan(b/a)$ for the larger acute angle of (a, b, c) , (a', b', c') for the composed triple with $a' = |3a - 4b|$, $b' = 4a + 3b$, and

$$\beta(a', b') := \arctan\left(\frac{b'}{a'}\right)$$

for the corresponding acute angle of the composed triangle (the larger one if $b' > a'$, the smaller one if $a' > b'$),

$$\cot\left(\frac{\alpha_0 + \beta(a', b')}{4}\right) = \varphi.$$

We prove this identity in full generality, unconditionally, as Proposition 3.5 in Section 3. (The condition $b/a < 7$ classifies whether $\beta(a', b')$ is the larger or smaller acute angle of the composed triple; the identity itself requires no such restriction.) We emphasize that $\beta(a', b')$ is the *ordinary* acute angle of triangle (a', b', c') and, despite its superficial resemblance, is *not* generally the same as the Euclid angle $\phi(u', v')$ of Definition 2.3 applied to the Euclid parameters of (a', b', c') ; see Remark 3.6 for the precise relationship.

This observation naturally raises the question whether the same construction can be extended to a transcendental target. The present author answered this question in the affirmative in an earlier preprint [3]. For every primitive Pythagorean triple, there exists an explicit, provably convergent sequence of partner triples such that

$$\cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{4}\right) \rightarrow \frac{\pi}{2}.$$

In the present paper, we go further. We show that the construction is *universal*: for any positive real constant $T > 0$, there exists an integer denominator D such that an infinite sequence of primitive Pythagorean triples satisfies

$$\cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{D}\right) \rightarrow T.$$

The sequence is generated explicitly by the continued fraction convergents (with a parity correction using semiconvergents) of the target ratio

$$t = \tan\left(\frac{D \arccot(T) - \alpha_0}{2}\right).$$

We prove that the convergence rate is $O(1/u_k^2)$ for subsequences of true convergents, and $O(1/u_k)$ when semiconvergents are used.

We demonstrate this universality with five pillar constants: π , Euler's number e , the Euler–Mascheroni constant γ , Apéry's constant $\zeta(3)$, and Catalan's constant G . For each, we provide explicit convergence tables.

The paper is organized as follows. Section 2 collects the necessary preliminaries on primitive Pythagorean triples, the Euclid parametrization, the formal definition of the Pythagorean angle lattice, and continued fractions [5]. Section 3 presents the generalised construction, proves the main theorem, and establishes the golden ratio and

metallic means as two exact special cases of the general framework. Section 4 provides numerical verification for the five pillar constants. Section 5 concludes with a summary and open problems.

2. Preliminaries

2.1 Primitive Pythagorean Triples and Euclid Parametrization

A primitive Pythagorean triple is an ordered triple (a, b, c) of positive integers satisfying

$$a^2 + b^2 = c^2, \quad \gcd(a, b, c) = 1, \quad a < b < c.$$

By the classical Euclid parametrization [4, 8], there exist coprime integers $u > v > 0$ of opposite parity such that

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2.$$

We call (u, v) the Euclid parameters of the triple.

Remark 2.1. In the numerical tables of Section 4, triples are listed in the order produced by the Euclid map $(u^2 - v^2, 2uv, u^2 + v^2)$. This may not satisfy the ordering $a < b < c$; the mathematical content is unaffected.

Definition 2.2 (Larger acute angle). For a primitive Pythagorean triple (a, b, c) , define

$$\alpha(a, b) = \arctan\left(\frac{\max(a, b)}{\min(a, b)}\right) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

This is always the larger of the two acute angles of the triangle, since $a \neq b$ for a primitive triple.

Definition 2.3 (Euclid angle map). For coprime $u > v > 0$ of opposite parity, define

$$\phi(u, v) = 2\arctan\left(\frac{v}{u}\right) \in \left(0, \frac{\pi}{2}\right).$$

Remark 2.4. As v/u ranges over $(0, 1)$, $\phi(u, v)$ ranges continuously and monotonically over $(0, \pi/2)$. The map $v/u \mapsto \phi(u, v)$ is a bijection from $(0, 1)$ onto $(0, \pi/2)$.

Important note: $\phi(u, v)$ equals the *larger* acute angle of the corresponding triple only when $v/u > \sqrt{2} - 1 \approx 0.4142$. When $v/u < \sqrt{2} - 1$, $\phi(u, v)$ is the *smaller* acute angle. Throughout this paper, we refer to $\phi(u, v)$ as the *Euclid angle* of the triple to avoid this ambiguity.

2.2 The Pythagorean Angle Lattice

Definition 2.5 (Pythagorean angle lattice). Let $\mathcal{T}$ be the set of all primitive Pythagorean triples with Euclid parameters (u, v) satisfying $\gcd(u, v) = 1$ and $u \not\equiv v \pmod{2}$. Define

$$\mathcal{L} = \left\{ \cot\left(\frac{\phi(u_0, v_0) + \phi(u_1, v_1) + \cdots + \phi(u_n, v_n)}{D}\right) : (u_i, v_i) \in \mathcal{T}, D \in \mathbb{N}, n \in \mathbb{N} \right\}.$$

This is the *Pythagorean angle lattice*.

In this paper, we fix one starting triple (a_0, b_0, c_0) with Euclid parameters (u_0, v_0) and consider the subset

$$\mathcal{L}_{\alpha_0} = \left\{ \cot\left(\frac{\alpha_0 + \phi(u, v)}{D}\right) : \gcd(u, v) = 1, u \not\equiv v \pmod{2}, D \in \mathbb{N} \right\}.$$

The constructive universality proven in this paper applies to $\mathcal{L}_{\alpha_0}$. The general multi-angle lattice $\mathcal{L}$ is left for future investigation.

2.3 Continued Fractions

Let $t \in (0, 1)$ be irrational with continued fraction expansion

$$t = [0; a_1, a_2, a_3, \dots].$$

Define $p_{-1} = 1, p_0 = 0, q_{-1} = 0, q_0 = 1$, and recursively

$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}.$$

The convergents p_k/q_k satisfy $\gcd(p_k, q_k) = 1$, $p_k/q_k \rightarrow t$, and

$$\left| t - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} = O\left(\frac{1}{q_k^2}\right).$$

For the theory of continued fractions, see [5].

2.4 The Parity Condition and Semiconvergents

For a pair (p, q) to serve as Euclid parameters (u, v) of a primitive Pythagorean triple, it must satisfy:

1. $\gcd(p, q) = 1$;
2. p and q have opposite parity (i.e., $p + q$ is odd).

Classical continued fractions guarantee coprimality but not opposite parity. The following lemma repairs this obstruction.

Lemma 2.6 (Admissible approximants exist infinitely often). There are infinitely many true convergents p_k/q_k with $p_k + q_k$ odd. Consequently, there are infinitely many coprime, opposite-parity pairs converging to t . Moreover, semiconvergents provide a practical method for finding admissible pairs with smaller denominators when a true convergent fails the parity test.

Proof. Let $S_n = p_n + q_n$. From the determinant identity $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, reducing modulo 2 gives

$$p_n q_{n-1} + p_{n-1} q_n \equiv 1 \pmod{2}.$$

If both S_n and S_{n-1} were even, then all four of $p_n, q_n, p_{n-1}, q_{n-1}$ would be odd (coprimality excludes them being all even), making the left-hand side $1 \cdot 1 + 1 \cdot 1 \equiv 0 \pmod{2}$, a contradiction. Hence S_n and S_{n-1} cannot both be even. Since a sequence of integers with no two consecutive evens must contain infinitely many odds, infinitely many S_n are odd. Therefore, infinitely many true convergents are admissible.

When a true convergent fails the parity test, semiconvergents

$$(p_{k-2} + mp_{k-1}, q_{k-2} + mq_{k-1}), \quad m = 1, \dots, a_k,$$

can be used to find an admissible pair with smaller denominator. This is the practical role of semiconvergents in the numerical examples of Section 4. For the error bound $O(1/u_k)$ for semiconvergents, see [5].

3. The Generalised Construction

3.1 The Main Theorem

Theorem 3.1 (Generalised π -thagoras). Let $T > 0$ be any positive real number. Let (a_0, b_0, c_0) be any primitive Pythagorean triple with larger acute angle α_0 . Then there exists an integer $D \in \mathbb{N}$ such that:

$$0 < \alpha^* = \text{Darccot}(T) - \alpha_0 < \frac{\pi}{2}.$$

Define

$$t = \tan\left(\frac{\alpha^*}{2}\right) \in (0, 1).$$

Let (u_k, v_k) be the sequence of admissible (coprime, opposite-parity) continued fraction convergents (or semiconvergents) of t guaranteed by Lemma 2.6, and let (a_k, b_k, c_k) be the corresponding primitive Pythagorean triples with Euclid angles $\phi(u_k, v_k)$. Then

$$\cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{D}\right) \rightarrow T \quad \text{as } k \rightarrow \infty.$$

Moreover, the error satisfies:

$$\left| \cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{D}\right) - T \right| = \begin{cases} O(1/u_k^2), & \text{if } (u_k, v_k) \text{ is a true convergent,} \\ O(1/u_k), & \text{if } (u_k, v_k) \text{ is a semiconvergent.} \end{cases}$$

Proof. Let $\theta = \operatorname{arccot}(T) = \arctan(1/T) \in (0, \pi/2)$. Since $\theta \in (0, \pi/2)$, the interval $(\alpha_0, \alpha_0 + \pi/2)$ has length $\pi/2 > \theta$. Therefore, the sequence $D\theta$ for $D = 1, 2, 3, \dots$ must land in this interval for some D . The smallest such D is given by $D = \lfloor \alpha_0/\theta \rfloor + 1$. Hence $\alpha^* = D\theta - \alpha_0 \in (0, \pi/2)$.

Since $t = \tan(\alpha^*/2) \in (0, 1)$, by Lemma 2.6 there exists a sequence of admissible Euclid pairs (u_k, v_k) with $v_k/u_k \rightarrow t$. The continuity of $\phi(u, v) = 2\arctan(v/u)$ gives

$$\phi(u_k, v_k) \rightarrow 2\arctan(t) = \alpha^*.$$

Therefore,

$$\frac{\alpha_0 + \phi(u_k, v_k)}{D} \rightarrow \frac{\alpha_0 + \alpha^*}{D} = \theta.$$

Since $\cot$ is continuous at θ and $\cot\theta = T$, we get the desired limit.

For the error bound, from continued fraction theory, for true convergents:

$$\left| \frac{v_k}{u_k} - t \right| = O\left(\frac{1}{u_k^2}\right).$$

For semiconvergents selected by Lemma 2.6, the bound is generally

$$\left| \frac{v_k}{u_k} - t \right| = O\left(\frac{1}{u_k}\right),$$

as the quadratic bound is not guaranteed for semiconvergents other than the terminal one in each block (see [5]).

Since $\arctan$ has bounded derivative on $[0, 1]$,

$$|\phi(u_k, v_k) - \alpha^*| = 2|\arctan(v_k/u_k) - \arctan(t)| = \begin{cases} O(1/u_k^2), & \text{for true convergents,} \\ O(1/u_k), & \text{for semiconvergents.} \end{cases}$$

The function $x \mapsto \cot((\alpha_0 + x)/D)$ is differentiable with bounded derivative in a neighbourhood of α^* (since $\theta \in (0, \pi/2)$ is bounded away from the singularities of $\cot$). Hence the mean value theorem gives the claimed bound. The implied constants depend on T and degrade as $T \rightarrow 0^+$ or $T \rightarrow \infty$.

Corollary 3.2. For every positive real constant T , and for every starting primitive Pythagorean triple, the construction above produces an explicit sequence of partner triples achieving arbitrarily many correct decimal digits of T . The number of correct digits grows linearly in k for subsequences of true convergents.

3.2 The Algorithmic Construction

The proof of Theorem 3.1 is constructive. The following procedure summarises the algorithm:

1. Input $T > 0$ and starting triple (a_0, b_0, c_0) .
2. Compute $\alpha_0 = \arctan(\max(a_0, b_0)/\min(a_0, b_0))$.
3. Set $\theta = \operatorname{arccot}(T) = \arctan(1/T)$.
4. Compute $D = \lfloor \alpha_0/\theta \rfloor + 1$.
5. Compute $\alpha^* = D\theta - \alpha_0$ and $t = \tan(\alpha^*/2)$.
6. Generate the continued fraction convergents p_k/q_k of t .
7. For each convergent p_k/q_k :
 - If $p_k + q_k$ is odd, set $(u_k, v_k) = (p_k, q_k)$.
 - Otherwise, search the semiconvergents $p_{k-2} + mp_{k-1}$, $q_{k-2} + mq_{k-1}$ for the first $m \in \{1, \dots, a_k\}$ with odd sum.
8. Compute $(a_k, b_k, c_k) = (u_k^2 - v_k^2, 2u_kv_k, u_k^2 + v_k^2)$.

9. Compute $\phi(u_k, v_k) = 2\arctan(v_k/u_k)$.
10. Output $\cot((\alpha_0 + \phi(u_k, v_k))/D)$.

The termination of each step is guaranteed by Lemma 2.6. The k -th admissible pair is found in $O(k)$ continued-fraction steps: since Lemma 2.6's argument shows no two consecutive convergents can both fail the parity test, at most half of the first $2k$ convergents can be inadmissible, so the k -th admissible pair occurs among the first $2k$ convergents, giving the $O(k)$ bound directly.

3.3 The Metallic Means as a Special Case

The metallic-mean identity from [2] is recovered as a special exact (non-limiting) case of the framework. When T is such that $t = \tan(\alpha^*/2)$ is a rational number exactly realizable by a single primitive triple, and $D = 4$, the construction terminates in one step.

Specifically, for a primitive triple (a, b, c) with $c - b \in \{1, 2, 8\}$, the governing parameter $n = 2a/(c - b)$ gives:

$$\cot\left(\frac{\theta}{4}\right) = \frac{n + \sqrt{n^2 + 4}}{2} = M_n.$$

This identity holds for all $n \geq 5$. The cases $n = 1, 2, 3, 4$ are exceptional and are handled separately in [2].

Thus, the framework contains the metallic means as exact algebraic points embedded within a continuum of transcendental limits.

3.4 The Golden Ratio as a Second Special Case

The introductory example involving the golden ratio φ (obtained by scaling a primitive triple by the fundamental $(3, 4, 5)$ triple via the Brahmagupta–Fibonacci identity) is not an isolated numerical curiosity: it is a genuine closed-form identity that holds for *every* primitive Pythagorean triple with $a < b < c$, with no restriction whatsoever. We state and prove this precisely here.

Definition 3.3 (Ratio angle). For positive reals x, y , define $\beta(x, y) = \arctan(y/x) \in (0, \pi/2)$. Note that for the starting triple, $\alpha_0 = \beta(a, b)$, since $b > a$.

Lemma 3.4 (A golden-ratio angle identity). Let $\varphi = (1 + \sqrt{5})/2$ and $\theta = \operatorname{arccot}\varphi$. Then

$$\tan(2\theta) = 2, \quad \tan(4\theta) = -\frac{4}{3}, \quad 4\operatorname{arccot}\varphi = \pi - \arctan\left(\frac{4}{3}\right).$$

Proof. Let $t = \tan\theta = 1/\varphi$. Since $\varphi^2 = \varphi + 1$, dividing by φ^2 gives $1 = 1/\varphi + 1/\varphi^2$, i.e. $t^2 + t - 1 = 0$, so $t^2 = 1 - t$. By the double-angle formula, $\tan(2\theta) = 2t/(1 - t^2) = 2t/t = 2$. Applying the double-angle formula again gives $\tan(4\theta) = 4/(-3) = -4/3$. Since $\theta \in (0, \pi/4)$, we have $2\theta \in (0, \pi/2)$ and $\tan(2\theta) = 2 > 0$, so $2\theta = \arctan 2$. Then $4\theta \in (0, \pi)$, and since $\tan(4\theta) < 0$, necessarily $4\theta \in (\pi/2, \pi)$, giving $4\theta = \pi - \arctan(4/3)$.

Proposition 3.5 (Golden ratio from Pythagorean angle composition). Let (a, b, c) be *any* primitive Pythagorean triple with $a < b < c$ and $\alpha_0 = \beta(a, b)$. Let

$$a' = |3a - 4b|, \quad b' = 4a + 3b,$$

so that (a', b', c') with $c' = \sqrt{a'^2 + b'^2}$ is the triple obtained by the Gaussian-integer product $(a + bi)(3 + 4i) = -a' + b'i$ (the sign of the real part is always negative, since $b/a > 1 > 3/4$). Then, with no restriction on (a, b, c) ,

$$\alpha_0 + \beta(a', b') = \pi - \arctan\left(\frac{4}{3}\right) = 4\operatorname{arccot}\varphi,$$

and therefore

$$\cot\left(\frac{\alpha_0 + \beta(a', b')}{4}\right) = \varphi.$$

Proof. Let $\psi_w = \arctan(4/3)$, the larger acute angle of (3,4,5). Since complex multiplication adds arguments,

$$\arg((a + bi)(3 + 4i)) = \arg(a + bi) + \arg(3 + 4i) = \alpha_0 + \psi_w.$$

Writing the product as $-a' + b'i$ with $a' = |3a - 4b| = 4b - 3a > 0$ (as $b/a > 1 > 3/4$) and $b' = 4a + 3b > 0$, this complex number lies in the second quadrant, so

$$\arg(-a' + b'i) = \pi - \arctan\left(\frac{b'}{a'}\right) = \pi - \beta(a', b').$$

Equating the two expressions for the argument,

$$\alpha_0 + \psi_w = \pi - \beta(a', b') \Rightarrow \alpha_0 + \beta(a', b') = \pi - \psi_w,$$

which by Lemma 3.4 equals $4\text{arccot}\varphi$. Taking cotangent of both sides after dividing by 4 gives the result. Note that this argument uses only elementary properties of complex multiplication and never invokes the Euclid parametrization of (a', b', c') ; in particular it requires no parity hypothesis on a, b, a' , or b' , and holds unconditionally.

Note that (a', b', c') need not be primitive: for a positive-density subset of starting triples, the pair (a', b', c') is a multiple of a primitive triple. This does not affect the identity above, since $\beta(a', b') = \beta(a'/d, b'/d)$ for any common divisor d .

Remark 3.6 (Relation to the Euclid angle – and the source of the original error). It is tempting to identify $\beta(a', b')$ with the Euclid angle $\phi(u', v')$ of Definition 2.3, computed from the (essentially unique) Euclid parameters (u', v') of (a', b', c') . This identification is only sometimes correct. Since $a' \equiv a \pmod{2}$ (because $a' = |3a - 4b|$ and $4b$ is even), we have:

- If a (equivalently a') is odd, then $a' = u'^2 - v'^2$, $b' = 2u'v'$, and $\tan\phi(u', v') = 2u'v'/(u'^2 - v'^2) = b'/a'$, so $\phi(u', v') = \beta(a', b')$ exactly.
- If a (equivalently a') is even, then $a' = 2u'v'$, $b' = u'^2 - v'^2$, so $\tan\phi(u', v') = a'/b'$, and $\phi(u', v') = \pi/2 - \beta(a', b')$ — the Euclid angle is the *complement* of the angle actually needed.

Consequently, $\cot((\alpha_0 + \phi(u', v'))/4) = \varphi$ holds only when a is odd; for triples with a even (e.g. (20,21,29), (8,15,17), (20,99,101)) this quantity is *not* φ at all (numerically, $\approx 4.016, 2.507, 1.727$ respectively for these three), even though the correct identity of Proposition 3.5, stated in terms of $\beta(a', b')$, holds for all three. This is why the present formulation avoids $\phi(u', v')$ entirely.

One further consequence: because $\pi/2 - \phi(u', v')$ corresponds to a pair with *equal* parity (both odd) rather than opposite parity, it is not itself an admissible Euclid angle in the sense of Definition 2.3. Hence, for a even, the golden-ratio identity, while true, is *not* realized as an element of the formally defined Pythagorean angle lattice $\mathcal{L}_{\alpha_0}$ (Definition 2.5) via the Euclid angle of a genuine admissible partner triple; it is a separate elementary consequence of complex-number argument addition. For a odd, by contrast, $\beta(a', b') = \phi(u', v')$ and the identity *is* a genuine element of $\mathcal{L}_{\alpha_0}$ provided (a', b', c') is primitive; if (a', b', c') is non-primitive, the reduced triple should be used to obtain genuine Euclid parameters, and the angle identity is unaffected after this reduction.

Numerical verification of Proposition 3.5 across primitive Pythagorean triples, with no restriction on b/a . All values independently confirmed to 50 significant digits; deviation from ϕ is at arbitrary-precision noise level ($< 10^{-49}$) in every case.

Triple (a, b, c)	b/a	Scaled triple (a', b', c')	$\cot((\alpha_0 + \beta(a', b'))/4)$
(3,4,5)	1.333	(7,24,25)	1.6180339887 ...
(20,21,29)	1.050	(24,143,145)	1.6180339887 ...
(8,15,17)	1.875	(36,77,85)	1.6180339887 ...
(5,12,13)	2.400	(33,56,65)	1.6180339887 ...
(20,99,101)	4.950	(336,377,505)	1.6180339887 ...
(9,40,41)	4.444	(133,156,205)	1.6180339887 ...
(11,60,61)	5.455	(207,224,305)	1.6180339887 ...
(13,84,85)	6.462	(297,304,425)	1.6180339887 ...
(15,112,113)	7.467	(403,396,565)	1.6180339887 ...

Corollary 3.7. Proposition 3.5 applied to the self-composition $(3,4,5) \times (3,4,5) = (7,24,25)$ recovers exactly the example stated informally in Section 1: $\alpha_0 = \arctan(4/3)$, $\beta(a', b') = \arctan(24/7)$ (which here coincides with the Euclid angle $\phi(4,3)$, since $a' = 7$ is odd), and $\cot((\alpha_0 + \beta(a', b'))/4) = \phi$.

3.5 Uniformity of the Approximation

The construction in Theorem 3.1 provides, for each T , a sequence converging to T . In fact, the construction is uniform on compact intervals: for any compact interval $[a, b] \subset (0, \infty)$, the error bounds in Theorem 3.1 hold uniformly in $T \in [a, b]$.

Since $[a, b]$ is compact and $T \mapsto \theta(T) = \operatorname{arccot}(T)$ is continuous, $\theta(T)$ ranges over a compact subset of $(0, \pi/2)$ as T ranges over $[a, b]$, staying bounded away from 0 and $\pi/2$. Hence the derivative of $x \mapsto \cot((\alpha_0 + x)/D)$ is uniformly bounded on this range, and the mean-value-theorem constant in the proof of Theorem 3.1 can be taken uniform over $T \in [a, b]$. The continued-fraction approximation bound $O(1/u_k^2)$ itself already carries a universal constant independent of T (specifically, the bound $|t - p_k/q_k| < 1/q_k^2$ is universal). Thus the combined error bound is uniform over $T \in [a, b]$.

4 Numerical Verification: Five Pillar Constants

In this section, we apply the Generalised π -thagoras Theorem (Theorem 3.1) to five cornerstone constants. In all cases, we use the canonical starting triple $(3,4,5)$, for which

$$\alpha_0 = \arctan\left(\frac{4}{3}\right) \approx 0.9272952180 \text{ rad } (53.1301^\circ).$$

The numerical values of the constants are taken from Finch [6]. All values were recomputed using arbitrary-precision arithmetic (verified to 50 significant digits).

Remark 4.1. All table entries have been independently verified to 50 significant digits using arbitrary-precision arithmetic, ensuring consistency with the theorem's claims.

4.1 Two Worked Examples in Full

4.1.1 Approximation of π

For $T = \pi$, we have $\theta = \operatorname{arccot}(\pi) = \arctan(1/\pi) \approx 0.3081690711$ rad. We choose $D = 4$, since

$$0 < 4\theta - \alpha_0 \approx 0.3053810665 < \frac{\pi}{2}.$$

Thus $\alpha^* = 4\theta - \alpha_0 \approx 0.3053810665$ rad, and

$$t = \tan\left(\frac{\alpha^*}{2}\right) \approx 0.1538883342 \dots$$

The continued fraction of t is

$$t = [0; 6, 2, 139, 1, 4, 1, 1, 2, 1, \dots].$$

The true convergents give pairs (6,1), (13,2), and (1813,279). However, (1813,279) has both entries odd, violating the opposite-parity condition. The semiconvergent repair yields the admissible pair (32,5) between 1/6 and 2/13, corresponding to $m = 2$.

Table 2 shows the convergence.

Convergence to π from the Pythagorean angle lattice with $D = 4$. All (u, v) pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot((\alpha_0 + \phi(u, v))/4)$	Error $(\cot - T)$
1	(6,1)	(35,12,37)	3.07518381	-6.64×10^{-2}
2	(13,2)	(165,52,173)	3.14181661	$+2.24 \times 10^{-4}$
3	(32,5)	(999,320,1049)	3.12910410	-1.25×10^{-2}

4.1.2 Approximation of the Euler–Mascheroni Constant γ

For $T = \gamma \approx 0.577215664901533$, we have:

$$\theta = \operatorname{arccot}(\gamma) = \arctan(1/\gamma) \approx 1.0472985103 \text{ rad } (60.0058^\circ).$$

We choose $D = 1$, since

$$0 < \theta - \alpha_0 \approx 0.1200032923 < \frac{\pi}{2}.$$

Thus $\alpha^* = \theta - \alpha_0 \approx 0.1200032923 \text{ rad}$, and

$$t = \tan\left(\frac{\alpha^*}{2}\right) \approx 0.0600737559 \dots$$

The true convergents of t are (16,1), (17,1) $\times$, (33,2), (50,3), (233,14), (283,17) $\times$, (1082,65), etc. (where $\times$ denotes parity-invalid). The pair (1082,65) is admissible and gives the third data point below.

Table 3 shows the convergence.

Convergence to the Euler–Mascheroni constant γ from the Pythagorean angle lattice with $D = 1$. All pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot(\alpha_0 + \phi(u, v))$	Error $(\cot - T)$
1	(16,1)	(255,32,257)	0.570788530	-6.43×10^{-3}
2	(50,3)	(2491,300,2509)	0.577411635	$+1.96 \times 10^{-4}$
3	(1082,65)	(1166499, 140660, 1174949)	0.577215183	-4.81×10^{-7}

4.2 Full Convergence Tables

4.2.1 Euler's Number e ($D = 3$)

For $T = e$, $\theta = \operatorname{arccot}(e) = \arctan(1/e) \approx 0.3525134218 \text{ rad}$. The target $t \approx 0.0652147404$ has continued fraction $[0; 15, 2, 1, 177, \dots]$. The true convergents give (15,1) $\times$, (31,2), and (46,3); (15,1) is invalid (both odd), but (31,2) and (46,3) are admissible.

Table 4 shows the convergence.

Convergence to Euler's number e from the Pythagorean angle lattice with $D = 3$. All pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot((\alpha_0 + \phi(u, v))/3)$	Error $(\cot - T)$
1	(31,2)	(957,124,965)	2.72217750	$+3.90 \times 10^{-3}$
2	(46,3)	(2107,276,2125)	2.71826707	-1.48×10^{-5}
3	(123,8)	(15065,1968,15193)	2.71925165	$+9.70 \times 10^{-4}$

4.2.2 Apéry's Constant $\zeta(3)$ ($D = 2$)

For $T = \zeta(3)$, $\theta = \operatorname{arccot}(\zeta(3)) = \arctan(1/\zeta(3)) \approx 0.6938961352$ rad. The target $t \approx 0.2344055288$ has continued fraction

$$t = [0; 4, 3, 1, 3, 7, 1, 2, 1, 2, 1, 1, \dots].$$

The true convergents are (4,1), (13,3)[×], (17,4), and (64,15); (13,3) is invalid (both odd), but (17,4) and (64,15) are admissible. Table 5 shows the convergence.

Convergence to Apéry's constant $\zeta(3)$ from the Pythagorean angle lattice with $D = 2$. All pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot((\alpha_0 + \phi(u, v))/2)$	Error $(\cot - T)$
1	(4,1)	(15,8,17)	1.16666667	-3.54×10^{-2}
2	(17,4)	(273,136,305)	1.20000000	-2.06×10^{-3}
3	(64,15)	(3871, 1920, 4321)	1.20212766	$+7.08 \times 10^{-5}$

4.2.3 Catalan's Constant G ($D = 2$)

For $T = G$, $\theta = \operatorname{arccot}(G) = \arctan(1/G) \approx 0.8292301520$ rad. The target $t \approx 0.3827898115$ has continued fraction

$$t = [0; 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 17, \dots].$$

The true convergents are (2,1), (3,1)[×], (5,2), (8,3), (13,5)[×], (34,13), etc.; (3,1) and (13,5) are invalid (both odd), but (5,2), (8,3), and (34,13) are admissible.

Table 6 shows the convergence.

Convergence to Catalan's constant G from the Pythagorean angle lattice with $D = 2$. All pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot((\alpha_0 + \phi(u, v))/2)$	Error $(\cot - T)$
1	(2,1)	(3,4,5)	0.750000000	-1.66×10^{-1}
2	(5,2)	(21,20,29)	0.888888889	-2.71×10^{-2}
3	(8,3)	(55,48,73)	0.928571429	$+1.26 \times 10^{-2}$

4.3 Summary Across All Five Constants

Table 7 summarises the construction for the five pillar constants.

Summary of the Pythagorean angle lattice construction for the five pillar constants.

Constant	Symbol	D	t	First valid (u, v)	Error (2nd approx)
Pi	π	4	0.1538883342...	(6,1)	$+2.24 \times 10^{-4}$
Euler's Number	e	3	0.0652147404...	(31,2)	-1.48×10^{-5}
Euler-Mascheroni	γ	1	0.0600737559...	(16,1)	$+1.96 \times 10^{-4}$
Apéry's	$\zeta(3)$	2	0.2344055288...	(4,1)	-2.06×10^{-3}
Catalan's	G	2	0.3827898115...	(2,1)	-2.71×10^{-2}

4.4 Discussion

The five constants chosen for this demonstration represent a complete spectrum of mathematical significance:

- π : the archetypal transcendental constant, proved transcendental by Lindemann (1882). Its appearance as the limit of the $D = 4$ construction demonstrates the power of the framework.
- e : the base of the natural logarithm, proved transcendental by Hermite (1873). Its convergence with $D = 3$ shows that the construction is not tied to a specific denominator.
- γ : the Euler-Mascheroni constant, ubiquitous in analytic number theory. Its irrationality remains an open problem. The fact that the Pythagorean angle

lattice contains a sequence converging to γ provides a new geometric characterisation of this elusive constant.

- $\zeta(3)$: Apéry's constant, proved irrational by Apéry (1979) [7]. Its inclusion demonstrates the connection to zeta values and special functions.
- G : Catalan's constant, arising in combinatorics and special functions. Its irrationality remains open, making it a compelling target for future investigation.

The fact that the same Pythagorean construction – varying only the denominator D and the target slope t – produces converging sequences for all five constants demonstrates the universality of the framework. The algebraic case (metallic means) and the transcendental cases emerge from the same geometric operation, with only the denominator D and target slope t varying. Since $v/u \mapsto \phi(u, v)$ is a bijection from $(0, 1)$ onto $(0, \pi/2)$, and rational slopes are dense in $(0, 1)$, the Euclid angles $\phi(u, v)$ are dense in $(0, \pi/2)$. The continued fraction construction provides an explicit, algorithmic way to navigate this density to reach any target angle.

5. Conclusion

We have introduced the *Pythagorean angle lattice* and proved that it is *constructively universal*: for every positive real constant $T > 0$, there exists an explicit sequence of primitive Pythagorean triples whose cotangent projections converge to T . The construction uses only the Euclid parametrization of primitive triples, continued fractions (with a novel semiconvergent parity correction), and elementary trigonometric identities. We demonstrated the universality of the framework with five pillar constants:

$$\pi, \quad e, \quad \gamma, \quad \zeta(3), \quad G.$$

For each, we provided explicit convergence tables, showing that the error decays as $O(1/u_k^2)$ for true convergents and $O(1/u_k)$ for semiconvergents. The framework unifies:

- Algebraic constants: metallic means and the golden ratio φ arise exactly (Propositions in Sections 3.3–3.4) rather than as limits.
- Transcendental constants: π and e arise as limits of the same construction.
- Analytic number theory: γ and $\zeta(3)$ are embedded in the lattice structure.
- Combinatorics: Catalan's constant G emerges naturally.

Open Problems

Several questions remain for future investigation:

1. Optimal error bounds. Can the $O(1/u_k^2)$ error for true convergents be sharpened, or is it optimal under the parity constraint? Within the parity-restricted sublattice, an argument analogous to Hurwitz's theorem suggests $O(1/u_k^2)$ is likely optimal, but a full proof remains open.
2. Extension to complex constants. Does the construction generalise to complex targets via complex continued fractions?
3. Spectral interpretation. The density of the Pythagorean angle lattice suggests a connection to the spectral theory of the modular surface; can this be made explicit?
4. Connection to the Langlands program. The semiconvergent parity correction arises from a 2-adic constraint; does this relate to deeper arithmetic structures?

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DJ Trigonometry A Square-Based Proportional Framework for Closed Geometric Figures

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Abstract

This research introduces DJ Trigonometry, a new and independent trigonometric framework developed to analyze angular and proportional relationships in closed and polygon-based geometric figures without reliance on circular geometry. Classical trigonometry is grounded in circles, radians, and trigonometric functions such as sine, cosine, and tangent, which often lead to conceptual abstraction and learning difficulties. To address these limitations, the proposed framework shifts the geometric foundation from circular to square-based structures. The core concepts of DJ Trigonometry include DJ Angle, DJ Degree, and DJ Measure, which together establish a direct proportional relationship between angular measurement and linear dimensions. A specialized measuring instrument, the DJ Protractor, is introduced to determine DJ Angles within polygonal geometries. Unlike classical angle systems, the DJ Degree represents a complete square-based rotational cycle and is geometrically distinct from circular rotation. A major theoretical contribution of this work is the DJ Badwan Theorem, which provides a direct proportional method for determining side lengths in a right-angled triangle by multiplying the adjacent side by the DJ Measure of the given acute angle. This approach eliminates the need for classical trigonometric functions and tables, relying on simple arithmetic and proportional reasoning. The framework is extended to determine hypotenuse relationships using the Pythagorean theorem within the DJ system. By redefining angular measurement through square-oriented geometry, DJ Trigonometry offers a simplified, intuitive, and pedagogically accessible alternative to classical trigonometry. The proposed system does not replace traditional methods but complements them by providing a new perspective for the study of closed geometric figures, with applications in mathematics education and conceptual geometry.

Keywords: DJ Trigonometry, DJ Measure, DJ Degree, DJ Angle, DJ Protractor, DJ Badwan Theorem.

1. Introduction

Trigonometry is a fundamental branch of mathematics that studies the relationship between angles and sides of geometric figures, especially triangles. Classical trigonometry is primarily based on circular geometry, where angles are measured using circular arcs and expressed in degrees or radians. Trigonometric functions such as sine, cosine, and tangent arise from this circular framework and are widely used in geometry, physics, astronomy, and engineering. Although classical trigonometry has strong theoretical foundations, it is often considered abstract and difficult for learners due to its dependence on circular reasoning, trigonometric functions, and symbolic representations. Furthermore, its geometric interpretation is largely limited to circular and rotational systems, making the analysis of polygonal and

closed figures less intuitive. To overcome these limitations, this research introduces DJ Trigonometry, a new and alternative trigonometric framework specifically developed for closed, polygon-based geometric figures such as triangles, squares, rectangles, cubes, and cuboids. Unlike conventional trigonometry, DJ Trigonometry does not rely on circular arcs, radians, or π . Instead, it is founded on square-based geometry and proportional relationships.

The central concept of this system is the DJ Measure, along with a newly defined angular unit known as the DJ Degree. These quantities provide a direct proportional link between angular measurement and linear dimensions in closed figures. To practically implement this measurement system, a specialized instrument called the DJ Protractor is introduced. This protractor is designed exclusively for measuring DJ Angles and visualizing angular relationships within polygonal geometries. A key theoretical contribution of DJ Trigonometry is the DJ Badwan Theorem, which establishes a direct proportional relationship between the sides of a right-angled triangle using the DJ Measure. This theorem enables the determination of one side of a triangle directly from another without employing classical trigonometric functions such as sine, cosine, or tangent. The DJ Badwan Theorem provides an independent and conceptually simple geometric interpretation of angle-side relationships within the DJ framework. By shifting the foundation of trigonometry from circular geometry to polygonal geometry, DJ Trigonometry offers a simplified, intuitive, and proportional approach to geometric analysis. This framework is particularly suited for educational and practical applications where clarity, direct measurement, and geometric intuition are essential. Thus, DJ Trigonometry presents a modern reinterpretation of trigonometric principles, offering a new perspective for understanding angular and linear relationships in closed geometric structures through the DJ Measure and the DJ Badwan Theorem.

2. Literature Review

Trigonometry has been a central area of mathematics since ancient times. In Euclid's Elements, the principles of proportion, similarity, and geometric construction were established, which later became the foundation of trigonometry. Classical trigonometry is primarily based on circular geometry, where angles are defined using arcs, radii, degrees, and radians.

In traditional trigonometry, functions such as sine, cosine, and tangent play a fundamental role. Historians of mathematics such as Boyer and Merzbach, Katz, and Maor have shown that this form of trigonometry evolved largely from astronomical calculations, circular motion, and wave-related phenomena. Standard textbooks such as Thomas' Calculus and Stewart's Calculus also present trigonometry predominantly within a circular framework. However, several scholars have pointed out that circle-based angle measurement systems often become abstract for learners, especially when trigonometry is applied to polygonal or closed geometric figures. Courant and Robbins emphasized that the more direct and visual a geometric interpretation is, the stronger conceptual understanding becomes. Similarly, Posamentier and Polya highlighted proportional reasoning as a powerful tool in the teaching and understanding of geometry. Although significant work has been carried out on polygonal geometry and square-based structures, attempts to redefine angular measurement and trigonometric relationships entirely within a non-circular framework remain limited. Traditional degree and radian systems are most effective for circles, cones, and spherical figures. However, no independent and simplified measurement system has been firmly

established for understanding angle-side relationships in polygons and closed linear figures. It is within this context that DJ Trigonometry offers a novel and independent perspective. This system is not dependent on circular arcs, π , or radians, but is instead grounded in square-based geometry and proportional relationships. The concepts of DJ Measure, DJ Degree, and DJ Angle are geometrically consistent while being fundamentally distinct from classical trigonometric definitions. In this sense, DJ Trigonometry practically fulfills the need for intuitive geometric interpretation emphasized by scholars such as Needham and Stillwell.

In particular, the DJ Badwan Theorem emerges as an independent proportional principle rather than a replacement for the classical tangent function. The theorem demonstrates that, in a right-angled triangle, the relationship between the sides can be established directly through the DJ Measure, enabling simplified and more intuitive calculations. According to the educational principles advocated by the National Council of Teachers of Mathematics (NCTM), such alternative frameworks enhance conceptual clarity and accessibility in mathematics learning. Thus, the existing literature indicates that while classical trigonometry is a mature and well-established discipline, there remains a clear need for an independent, non-circular trigonometric framework suitable for closed and polygon-based geometric figures. DJ Trigonometry addresses this research gap and contributes a new conceptual approach to the field of trigonometric analysis.

3. The DJ Trigonometric Framework

3.1 Measurement of DJ Angles Using the DJ Protractor

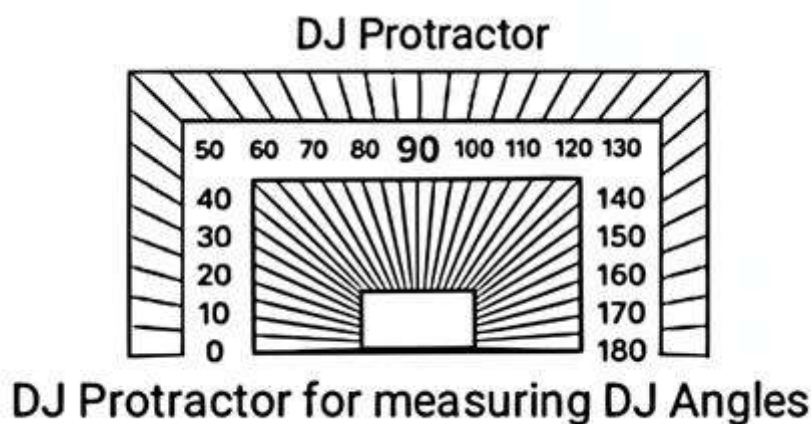


Figure 1. DJ Protractor for Measuring DJ Angles

The DJ Protractor is a specially proposed geometric instrument designed for measuring DJ Angles as defined within the DJ Trigonometric framework. This instrument differs from the traditional circular protractor because it is not based on circular geometry but on square-based geometry. The DJ Protractor is constructed in accordance with the concepts of DJ Degree and DJ Measure, enabling the representation of angular positions without the use of circular arcs, radians, or classical trigonometric functions. The structural design of the DJ Protractor is founded on the principle of square-based rotation, in which a complete square rotation is considered to be divided into 360 DJ Degrees. This structure enables consistent and proportional measurement of DJ Angles in closed and polygonal geometric figures and remains fully compatible with the theoretical framework of DJ Trigonometry. The method of using the DJ Protractor is defined as follows. The point at which the DJ Angle is to be measured is

placed at the center of the DJ Protractor. One arm of the DJ Angle is aligned with a selected standard radius, referred to as the reference radius. Subsequently, the second arm of the DJ Angle extends toward a specific radius or a DJ Degree marking indicated on the protractor. This marking represents the DJ Degree value of the given DJ Angle. From the obtained DJ Degree value, the corresponding DJ Measure can be directly determined using the conversion relationships defined within the DJ Trigonometric system.

Thus, the DJ Protractor may be regarded not only as an instrument for measuring angles but also as a means of clearly establishing the proportional relationship between a DJ Angle and its arms. This proportional representation forms the basis for further geometric and trigonometric calculations within DJ Trigonometry.

3.2 DJ Angle

A DJ Angle is an angular quantity defined within the DJ Trigonometric framework in accordance with square-based geometry. It represents the relative positional orientation of two arms about a point; however, unlike classical angles, its representation is not based on circular geometry but on the principle of square-based rotation. The measurement of a DJ Angle is expressed in terms of DJ Degree or DJ Measure, both of which are independently defined within DJ Trigonometry. A complete square-based rotation is considered to be divided into 360 DJ Degrees, and the measure of a DJ Angle is determined accordingly. This measurement system is not mathematically equivalent to the classical degree or radian systems, but instead represents an independent angular framework. The determination of a DJ Angle is carried out using the DJ Protractor, a specialized geometric instrument developed in accordance with square-based structural principles. The value of a DJ Angle can be further converted into DJ Measure, which is used to establish proportional relationships among the sides of right-angled triangles and other closed geometric figures. Thus, the DJ Angle constitutes a fundamental concept of DJ Trigonometry, providing a proportional representation of angular position and offering an alternative geometric perspective distinct from classical trigonometric approaches.

3.3 DJ Degree Measure

A DJ Degree is an angular unit of measurement used within the DJ Trigonometric framework for measuring DJ Angles. This unit is based on square-based rotational geometry and is distinct from the traditional circular degree system. In DJ Trigonometry, a complete square-based rotation is considered to be divided into 360 equal parts, each of which is defined as one DJ Degree. Accordingly, the DJ Degree provides a numerical representation of the square-based rotational position of a DJ Angle. It is explicitly emphasized that the DJ Degree is not used as a mathematical equivalent of the classical degree or radian measurement systems. Although the numerical value of 360 is adopted, the underlying geometric foundation is square-based rather than circular. Therefore, the meaning, representation, and application of the DJ Degree are fundamentally different from those of conventional angular units.

The DJ Degree is employed in the measurement of DJ Angles using the DJ Protractor and is further converted into DJ Measure to establish proportional relationships among the sides of right-angled triangles and other closed geometric figures.

Thus, the DJ Degree constitutes a fundamental angular unit of DJ Trigonometry, providing an independent and consistent framework for angular measurement within

square-based geometry. If the rotation from the initial side to the terminal side is one three-hundred-sixtieth ($1/360$) of a complete revolution, then the DJ Angle is said to have a measure of one DJ Degree, written as 1 DJ Degree. One DJ Degree is divided into 60 minutes, and one minute is further divided into 60 seconds. One sixtieth of a DJ Degree is called a minute and is written as $1'$, and one sixtieth of a minute is called a second and is written as $1''$.

Thus, $1 \text{ DJ Degree} = 60'$ and $1' = 60''$.

3.4 DJ Measure

A DJ Measure is a proportional quantity defined within the DJ Trigonometric framework that expresses the magnitude of a DJ Angle in relation to the linear dimensions of a geometric configuration. Unlike classical trigonometric ratios such as sine, cosine, or tangent, DJ Measure is derived from square-based rotational geometry and does not rely on circular arcs or traditional trigonometric functions. In DJ Trigonometry, the square-based rotational system divides a complete rotation into 360 DJ Degrees. The DJ Measure of an angle is obtained by relating the position of the angle's terminal arm, as determined using the DJ Protractor, to the corresponding DJ Degree value. This DJ Degree value is then translated into a DJ Measure through the defined conversion relationships within the DJ system. Mathematically, DJ Measure represents a ratio between the effective proportional extension of a geometric side relative to a standard reference within a square-based structure. It serves as a direct proportional parameter for establishing relationships among the sides of right-angled triangles and other closed geometric figures without resorting to classical trigonometric ratios or tables. The use of DJ Measure enables the representation of angular magnitude and its associated linear relationships in a form that is consistent with the square-oriented conceptual basis of DJ Trigonometry. It provides a coherent framework for analyzing geometric configurations where proportional reasoning replaces circular trigonometric functions. Thus, DJ Measure functions as a fundamental component of DJ Trigonometry, facilitating the translation of angular orientations into quantifiable proportional relationships suitable for geometric analysis and application.

Radius of a Square

If the side length of a square is a , then the radius r of the square is given by

$$r = a / 2$$

$$\text{Perimeter of the square} = (\text{DJ Measure}) \times r.$$

Another unit for measuring a DJ Angle is the DJ Measure. In a unit square of radius 1 unit, if a segment of the square perimeter of length 1 unit subtends a DJ Angle at the centre, then the measure of that DJ Angle is defined as 1 DJ. We know that the equivalent complete rotational boundary of a square of radius 1 unit is 8 units. Therefore, one complete rotation of the initial side produces a DJ Angle of 8 DJ. More generally, if a square has a radius of r units, then a segment of the square perimeter of length r units subtends a DJ Angle of 1 DJ at the centre. Equal segments of the square perimeter subtend equal DJ Angles. Hence, if in a square of radius r , a segment of the square perimeter of length l units will subtend a DJ Angle of l/r DJ at the centre. Thus, if in a square of radius r , a segment of the square perimeter of length l subtends a DJ Angle θ DJ at the centre, then the relation is given by $\theta = l/r$ in DJ Measure.

$$\text{DJ Measure} = (\text{Perimeter of the square}) / (\text{radius of the square})$$

$$\text{DJ Measure} = (\text{Perimeter of the square}) / (\text{radius of the square})$$

3.5 Relation between DJ Degree and DJ Measure

Since a square subtends a DJ Angle at the centre whose DJ Measure is 8, and its DJ Degree Measure is 360 DJ Degree, it follows that:

$$8 \text{ DJ} = 360 \text{ DJ Degree}$$

or

$$4 \text{ DJ} = 180 \text{ DJ Degree}$$

The above relation enables us to express a DJ Measure in terms of DJ Degree Measure and a DJ Degree Measure in terms of DJ Measure.

Using this relation, we obtain:

$$\begin{aligned} 1 \text{ DJ} &= 180 / 4 \text{ DJ Degree} \\ &= \text{approximately } 45 \text{ DJ Degree, Also,} \\ 1 \text{ DJ Degree} &= 4 / 180 \text{ DJ} \\ &= \text{approximately } 0.022 \text{ DJ} \end{aligned}$$

Thus, the relation between DJ Degree Measures and DJ Measures of some common DJ Angles is given in the following table:

DJ Degree						
30 DJ Degree	45 DJ Degree	60 DJ Degree	90 DJ Degree	180 DJ Degree	270 DJ Degree	360 DJ Degree
DJ Measure						
4/6	4/4	4/3	4/2	4	6	8

Table 1. Relation between DJ Degree and DJ Measure.

Notational Convention.

Since DJ Angles are measured either in DJ Degrees or in DJ Measure, we adopt the following notational convention. Whenever we write a DJ Angle θ DJ Degree, it means the DJ Angle whose DJ Degree Measure is θ . Similarly, whenever we write a DJ Angle β , it means the DJ Angle whose DJ Measure is β . It is important to note that when a DJ Angle is expressed in DJ Measure, the term "DJ Measure" is usually omitted and is understood from the context. For example, the quantities 4 and 4/4 are often written without explicitly mentioning "DJ Measure", and with this understanding,

$$4 \text{ DJ Measure} = 180 \text{ DJ Degree}$$

and

$$4/4 \text{ DJ Measure} = 45 \text{ DJ Degree.}$$

Hence, we obtain the following conversion relations:

$$\text{DJ Measure} = (4 / 180) \times \text{DJ Degree Measure}$$

$$\text{DJ Degree Measure} = (180 / 4) \times \text{DJ Measure}$$

$\text{DJ Measure} = (4 / 180) \times \text{DJ Degree Measure}$
$\text{DJ Degree Measure} = (180 / 4) \times \text{DJ Measure}$

Thus, using the above relations, one can easily convert the DJ Degree Measure of any DJ Angle into its DJ Measure, and vice versa.

3.6 DJ Acute Angle

A DJ acute Angle is defined as any DJ Angle less than 90 DJ Degree. It represents

an angle smaller than a right angle within the DJ measurement system. The DJ acute angle is composed of two fundamental parts, namely the D acute angle and the J acute angle, which together form the complete structure of the right-angled relationship in DJ trigonometry.

3.6.1 D Acute Angle

A D acute Angle is defined as 45 DJ Degree or any DJ Angle less than 45 DJ Degree. This type of angle represents the smaller range of acute measurements within the DJ system, focusing on sharp, narrow angular formations. The D acute angle symbolizes the lower spectrum of acute measurements and plays a significant role in the proportional and symmetrical understanding of closed geometrical figures.

3.6.2 J Acute Angle

A J acute angle is defined as 45 DJ Degree or any DJ Angle greater than 45 DJ Degree but less than 90 DJ Degree. This represents the broader portion of the acute range, where the angle extends closer to the right angle while maintaining its acute nature. The J acute angle thus complements the D acute angle to complete the total range of acute measurements within the DJ system.

Sum of DJ Angles in a Triangle

According to DJ Trigonometry, the sum of the three DJ Angles present in any triangle is equal to 180 DJ Degree. If the triangle is a right DJ Angle triangle, then one of its angles is a right DJ Angle, whose measure is 90 DJ Degree. Therefore, in a right DJ Angle triangle, the sum of the remaining two angles, namely the D acute angle and the J acute angle, is obtained by subtracting the right angle from 180 DJ Degree. Thus,

$$\text{D acute angle} + \text{J acute angle} = 180 \text{ DJ Degree} - 90 \text{ DJ Degree}$$

That is,

$$\text{D acute angle} + \text{J acute angle} = 90 \text{ DJ Degree}$$

$\text{D acute angle} + \text{J acute angle} = 90 \text{ DJ Degree}$
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Hence, it is established that in any right DJ Angle triangle, the sum of the D acute Angle and the J acute angle is always equal to 90 DJ Degree.

4. DJ Badwan Theorem

4.1 Statement of the Theorem

DJ Badwan Theorem states that within the DJ Trigonometric Framework, the relationship between the sides of a right-angled triangle corresponding to a given DJ acute angle is governed by direct proportionality rather than circular trigonometric functions. In a right-angled triangle, if the DJ Measure of a given D acute angle is known, then the length of the side opposite to that angle can be obtained directly by multiplying the length of the side adjacent to the angle by the DJ Measure of the angle. The essential requirement in DJ Trigonometry is that: The side adjacent to the D Acute Angle is considered as the radius of the square, and the side adjacent to the J Acute Angle is considered as the perimeter-related proportional side of the square. Hence, in DJ Trigonometry, the determination of triangle sides is governed exclusively by their adjacency to the D Acute Angle and the J Acute Angle, rather than by conventional geometric orientation. This square-based interpretation allows DJ Trigonometry to treat the triangle as a closed geometric figure, where angle-side relationships are expressed in a direct, proportional, and non-circular manner. As a result, the framework eliminates the necessity of classical trigonometric notions such as sine, cosine, tangent, and circular geometry, and instead relies on simple linear proportionality derived from

square-oriented geometry. This theorem establishes a simple, linear, and square-based proportional rule for determining side lengths without the use of sine, cosine, tangent, radians, or circular geometry.

4.2 Mathematical Formulation

Let the DJ Measure of a D acute angle be x . The DJ Measure is defined as the ratio of the Perimeter of the square to the radius of the imaginary square.

$$\text{DJ Measure} = (\text{Perimeter of the square}) / (\text{radius of the square})$$

Let O be the length of the side adjacent to the D acute angle. This side is also interpreted as the radius of the imaginary square.

Let P be the length of the side opposite to the D acute angle. This side is also interpreted as the perimeter of the imaginary square.

According to the DJ Badwan Theorem: in a right-angled triangle, if the D acute angle is given, then the length of the side opposite to the D acute angle (P), or equivalently the perimeter of the imaginary square, can be obtained by multiplying the length of the side adjacent to the D acute angle (O), or equivalently the radius of the imaginary square, by the DJ Measure (x) of that angle. Mathematically,

$$(x) = (4/180) \text{ D acute angle}$$

Furthermore, the value of P is always less than or equal to the value of O

$$\text{DJ Measure of the D acute angle} = x$$

$$\text{DJ Measure} \times (\text{radius of the square}) = (\text{Perimeter of the square})$$

$$\text{Length of the adjacent side of the D acute angle} = O = (\text{radius of the square})$$

$$\text{Length of the side opposite to the D acute angle} = P = (\text{Perimeter of the square})$$

$$P = O \times x$$

$$O = P/x$$

$P = O \times x$
$O = P/x$

Where:

= the side opposite the given acute angle

→ It can be the base or the perpendicular, depending on the position.

= the side adjacent to the given acute angle

→ It can be the base or the perpendicular, depending on the position.

The result obtained from this multiplication represents the actual length of the side opposite to the D acute angle. In other words, when the adjacent side is multiplied by the DJ Measure of the D acute angle, the obtained value is equal to the length of the side opposite to that angle.

4.3 Proof / Logical Justification

In DJ Trigonometry, angular measurement is not derived from circular arcs but from square-based geometry. The DJ Measure is defined as the ratio of the effective square boundary length to its corresponding radius. Consider a right-angled triangle in which the adjacent side to the D acute angle is treated as the radius of an imaginary square. The opposite side is interpreted as the proportional square boundary corresponding to that angle. Because equal DJ Angles subtend proportional square boundary lengths, the ratio of the opposite side to the adjacent side must be equal to the DJ Measure of the angle.

4.4 Comparative Clarification of DJ Trigonometry

It is necessary to explicitly clarify that all concepts used in DJ Trigonometry are not equivalent to the corresponding concepts of classical or traditional trigonometry. Rather, they are independent, distinct, and founded on a new geometric basis.

DJ Angle – The DJ Angle is not equivalent to the classical angle. Classical angles are fundamentally based on circular geometry, whereas the DJ Angle is defined within a square-based geometric framework. Consequently, their measurement, interpretation, and application differ at a fundamental level.

DJ Degree – Although the DJ Degree visually resembles the classical degree system by adopting the number 360, it is not mathematically equivalent. Classical degrees arise from circular rotation, while DJ Degree are derived from a square-based rotational cycle. Therefore, the geometric meaning of a DJ Degree is fundamentally different from that of a classical degree.

DJ Measure – The DJ Measure is not a reformulation of the radian measure. In classical trigonometry, the radian is defined as the ratio of arc length to radius in a circle. In contrast, the DJ Measure is derived from proportional relationships involving the perimeter and linear dimensions of a square. Hence, DJ Measure represents a polygon-based proportional quantity rather than a circular one.

DJ Protractor – The DJ Protractor is not a modified version of the traditional protractor. It is a newly designed geometric instrument developed specifically for measuring DJ Angles. While the classical protractor operates on circular angle measurement, the DJ Protractor functions according to square-based and polygonal geometry.

DJ Acute Angle – The DJ Acute Angle is conceptually different from the classical acute angle. In DJ Trigonometry, an acute angle is defined within a square-based framework and is further divided into two components: the D acute angle and the J acute angle. Such a structural division does not exist in classical trigonometry and arises naturally from the DJ geometric system.

DJ Badwan Theorem – The DJ Badwan Theorem is not equivalent to the classical trigonometric function $\tan \theta$. Although both involve a proportional relationship between sides of a right-angled triangle, their conceptual foundations and definitions are fundamentally different. The DJ Badwan Theorem is entirely based on DJ Degree and DJ Measure, which are independent of circular geometry and trigonometric functions. It employs direct proportional multiplication rather than functional evaluation.

DJ Formulae – All formulae used in DJ Trigonometry are distinct from those of classical trigonometry. In the DJ framework, relationships between angles and sides are expressed through linear proportional relationships instead of sine, cosine, or tangent functions.

Therefore, it is established that DJ Trigonometry is neither a transformation nor a replacement of classical trigonometry. Instead, it constitutes an independent, square-based, non-circular trigonometric framework designed to analyze angular and linear relationships in closed geometric figures such as triangles, squares, rectangles, and other polygonal structures.

5. Derived Relationships

5.1 Hypotenuse-Adjacent Side Relationship

Conceptual Proportional Relationship between the Hypotenuse (K) and the Adjacent Side (O) in DJ Trigonometry. By Pythagoras theorem, in a right-angled triangle,

the square of the hypotenuse is equal to the sum of the squares of the base and the perpendicular. Hence,

$$K^2 = O^2 + P^2$$

Here,

K denotes the hypotenuse of the right-angled triangle.

O denotes the base, which is the adjacent side to the given D acute angle.

P denotes the perpendicular, which is the opposite side to the same D acute angle.

Thus, in DJ Trigonometry, O is always taken as the side adjacent to the D acute angle, and P is always taken as the side opposite to the D acute angle.

According to DJ Trigonometry, the DJ Measure is defined as the ratio of the opposite side to the adjacent side corresponding to the D acute angle. Therefore,

$$\text{DJ Measure} = x = P / O$$

Further, the DJ Measure is related to the DJ Degree through the D acute angle as,

$$\text{DJ Measure} = (4 / 180 \text{ DJ Degree}) \times (\text{D acute angle})$$

Now, dividing the Pythagoras equation by O^2 , we obtain,

$$K^2 / O^2 = O^2 / O^2 + P^2 / O^2$$

$$K^2 / O^2 = 1 + (P / O)^2$$

$$\text{Since } P / O = \text{DJ Measure,}$$

$$K^2 / O^2 = 1 + (\text{DJ Measure})^2$$

Taking square root on both sides,

$$K / O = \sqrt{1 + (\text{DJ Measure})^2}$$

Hence, the relationship between the hypotenuse and the adjacent side (base) corresponding to the D acute angle is,

$$K = O \sqrt{1 + (\text{DJ Measure})^2}$$

From this expression, the adjacent side (base) can be written in terms of hypotenuse and DJ Measure as,

$$O = K / \sqrt{1 + (\text{DJ Measure})^2}$$

$K = O \sqrt{1 + (\text{DJ Measure})^2}$
$O = K / \sqrt{1 + (\text{DJ Measure})^2}$

5.2 Hypotenuse–Opposite Side Relationship

Conceptual Proportional Relationship between the Hypotenuse (K) and the Opposite Side (P) in DJ Trigonometry. Now again starting from the Pythagoras theorem,

$$K^2 = O^2 + P^2$$

Dividing the entire equation by P^2 ,

$$K^2 / P^2 = O^2 / P^2 + P^2 / P^2$$

$$K^2 / P^2 = O^2 / P^2 + 1$$

Since DJ Measure = P / O , it follows that,

$$O / P = 1 / (\text{DJ Measure})$$

Substituting this value,

$$K^2 / P^2 = 1 / (\text{DJ Measure})^2 + 1$$

Taking square root on both sides,

$$K / P = \sqrt{1 + 1 / (\text{DJ Measure})^2}$$

Thus, the relationship between the hypotenuse and the perpendicular (opposite side to the D acute angle) is,

$$K = P \sqrt{(1 + 1 / (\text{DJ Measure})^2)}$$

Rearranging the above expression, the perpendicular can be expressed as,

$$P = K (\text{DJ Measure}) / \sqrt{(1 + (\text{DJ Measure})^2)}$$

$$K = P \sqrt{(1 + 1 / (\text{DJ Measure})^2)}$$

$$P = K (\text{DJ Measure}) / \sqrt{(1 + (\text{DJ Measure})^2)}$$

6. Applications / Examples

6.1 Real-life Example 1

(i) Example: Determination of the Height of a Minaret Using DJ Trigonometry.

Suppose the height of a minaret standing on a level ground is to be determined. A suitable point is selected on the ground from where the top of the minaret is clearly visible. In this situation, the minaret, the ground, and the line of sight together form a right-angled triangle. At the selected point on the ground, the DJ Angle is measured using a DJ Protractor. The base of the DJ Protractor is aligned with the horizontal ground line, and its line of sight is directed toward the top of the minaret. The angular value obtained on the DJ Protractor represents the DJ Angle for this observation.

Let the measured DJ Angle be 30 DJ Degree. This value is used for further calculations. Assume that the horizontal distance from the observation point to the base of the minaret, that is, the adjacent side of the right-angled triangle, is

$$O = 15 \text{ meters.}$$

The height of the minaret corresponds to the opposite side P of the triangle. According to DJ Trigonometry, the DJ Measure corresponding to a given DJ Angle is obtained using the relation:

$$\text{DJ Measure} = (1/45 \text{ DJ Degree}) \times \text{DJ Angle}$$

Substituting the given value,

$$\text{DJ Measure} = (1/45 \text{ DJ Degree}) \times 30 \text{ DJ Degree} = 2/3$$

According to the DJ Badwan principle, the relationship between the opposite side and the adjacent side of a right-angled triangle is given by:

$$P = O \times \text{DJ Measure}$$

Substituting the known values,

$$P = 15 \times 2/3 = 30/3 \approx 10 \text{ meters}$$

Thus, using the DJ Angle measured with the DJ Protractor and the corresponding DJ Measure, the height of the minaret is found to be approximately 10 meters. Throughout this calculation, no trigonometric functions such as sine, cosine, or tangent, and no circular concepts such as radians or π are used. The result is obtained solely through DJ Angle, DJ Degree, and DJ Measure using simple arithmetic operations.

This example clearly demonstrates that DJ Trigonometry provides a simple, direct, and practical method for determining the height of structures such as minarets, buildings, poles, or trees in real-world situations.

6.2 Real-life Example 2

(ii) Example: Determination of the Hypotenuse (K) Using DJ Trigonometry.

Suppose a right-angled triangle is formed on a level ground. From a fixed point, a DJ Angle is made toward an object, and the length of the adjacent side corresponding to this angle is known. The objective is to determine the hypotenuse of the triangle.

First, the DJ Angle is measured using a DJ Protractor from the observation point toward the object. Let the measured angle be

DJ Angle = 30 DJ Degree.

Now assume that:

Adjacent side (O) = 15 meters.

According to DJ Trigonometry, the DJ Measure is first obtained from the given DJ Angle.

$$\text{DJ Measure} = (1/45 \text{ DJ Degree}) \times \text{DJ Angle}$$

$$\text{DJ Measure} = (1/45 \text{ DJ Degree}) \times 30 \text{ DJ Degree} = 2/3$$

Next, the relationship between the hypotenuse and the adjacent side in DJ Trigonometry is applied:

$$K = O \sqrt{(1 + (\text{DJ Measure})^2)}$$

Substituting the known values,

$$K = 15 \sqrt{(1 + (2/3)^2)}$$

$$K = 15 \sqrt{(1 + 4/9)}$$

$$K = 15 \sqrt{(13/9)}$$

$$K = 15 \times (\sqrt{13} / 3)$$

$$K = 5\sqrt{13} \approx 18.03 \text{ meters}$$

Thus, according to DJ Trigonometry, the hypotenuse of the right-angled triangle is approximately 18.03 meters. In this example, the hypotenuse is calculated using only the DJ Angle, the corresponding DJ Measure, and the hypotenuse-adjacent side relationship formula. No trigonometric functions such as sine, cosine, or tangent, and no circular concepts such as radians or π are used. The result is obtained solely through basic arithmetic operations and square roots. This example clearly demonstrates that the hypotenuse-adjacent side relationship in DJ Trigonometry provides a simple, direct, and effective method for practical calculations.

6.3 Real-life Example 3

(iii) Example: Determination of the Hypotenuse (K) Using DJ Trigonometry.

(Application of the Hypotenuse-Opposite Side Relationship)

Suppose a right-angled triangle is formed. The length of the opposite side (P) corresponding to a given DJ Angle is known, and the objective is to determine the hypotenuse (K) of the triangle. First, the DJ Angle toward the object is measured using a DJ Protractor. Let the measured angle be

DJ Angle = 30 DJ Degree.

Now assume that:

Opposite side (P) = 10 meters.

According to DJ Trigonometry, the DJ Measure is first obtained from the given DJ Angle.

$$\text{DJ Measure} = (1/45 \text{ DJ Degree}) \times \text{DJ Angle}$$

$$\text{DJ Measure} = (1/45 \text{ DJ Degree}) \times 30 \text{ DJ Degree} = 2/3$$

Next, the relationship between the hypotenuse and the opposite side in DJ Trigonometry is applied:

$$K = P \sqrt{(1 + 1 / (\text{DJ Measure})^2)}$$

Substituting the known values,

$$K = 10 \sqrt{(1 + 1 / (2/3)^2)}$$

$$K = 10 \sqrt{(1 + 1 / (4/9))}$$

$$K = 10 \sqrt{(1 + 9/4)}$$

$$K = 10 \sqrt{(13/4)}$$

$$K = 10 \times (\sqrt{13} / 2)$$

$$K = 5\sqrt{13} \approx 18.03 \text{ meters}$$

Thus, according to DJ Trigonometry, the hypotenuse of the right-angled triangle

is approximately 18.03 meters.

In this example, the hypotenuse is calculated using only the DJ Angle, the corresponding DJ Measure, and the hypotenuse–opposite side relationship formula. No trigonometric functions such as sine, cosine, or tangent, and no circular concepts such as radians or π are used. The result is obtained solely through basic algebraic operations and square-root evaluation.

This example clearly demonstrates that the hypotenuse–opposite side relationship in DJ Trigonometry provides a simple, direct, and effective approach for practical calculations.

6.4 Real-life Example 4

Formulae Used in Example 4

Radius of a square

If the side length of a square is a , then the radius r of the square is given by

$$r = a / 2$$

Relationship between DJ Degree and DJ Measure

The conversion from DJ Degree to DJ Measure is given by

$$\text{DJ Measure} = (4 / 180) \times \text{DJ Degree}$$

Definition of DJ Measure

DJ Measure is defined as the ratio of the square perimeter segment to the radius of the square

$$\text{DJ Measure} = \text{Length of square perimeter segment} / r$$

Formula for square perimeter segment corresponding to a DJ Angle

$$\text{Length of square perimeter segment} = r \times \text{DJ Measure}$$

Application for a given DJ Angle

If DJ Angle = 120 DJ Degree, then

$$\text{DJ Measure} = (4 / 180) \times 120$$

$$\text{DJ Measure} = 8 / 3$$

Application for a given radius

If radius $r = 3$ units, then

$$\text{Length of square perimeter segment} = 3 \times (8 / 3)$$

$$\text{Length of square perimeter segment} = 8 \text{ units}$$

Conceptual Note

All the above formulae are based on square-oriented geometry. No circular arc, no π , no radian measure, and no classical trigonometric functions such as sine, cosine, or tangent are used. The DJ Measure directly connects angular information with linear perimeter length through simple proportional relations. This perimeter segment represents the proportional square boundary length corresponding to the given DJ Angle, not a single physical side of the square.

Importance of DJ Trigonometry in Mobile Calculators

In the present digital era, mobile calculators have become a common and easily accessible tool for mathematical computations. However, in classical trigonometry, the use of mobile calculators requires trigonometric functions such as sine, cosine, and tangent, along with careful handling of degree–radian modes and circular concepts. This often makes the calculation process complex and confusing, especially for beginners. In contrast, DJ Trigonometry provides a simple, direct, and calculator-friendly framework.

In DJ Trigonometry, angles are defined exclusively as DJ Angles, and their measurement is expressed in DJ Degree. From a given DJ Angle, the DJ Measure is

obtained directly. The DJ Measure is a proportional quantity that establishes a direct relationship between the sides of a right-angled triangle. This process does not involve circular angles, radians, or the constant π . A key feature of DJ Trigonometry is that the entire trigonometric analysis is based on four fundamental formulas expressed solely in terms of the DJ Measure. These formulas require only basic algebraic operations such as multiplication, division, and square roots. Consequently, there is no need to select or evaluate any trigonometric functions on a mobile calculator. Thus, in DJ Trigonometry, the mobile calculator functions merely as a simple numerical computation device, while the DJ Angle and DJ Measure provide a clear and controlled representation of angular and proportional relationships. This significantly simplifies the calculation process and reduces the possibility of conceptual or operational errors. Therefore, it can be concluded that DJ Trigonometry, particularly through the use of DJ Angle, is highly important, practical, and educationally effective for mobile calculator-based trigonometric calculations.

7. Objectives

The objectives of the present research are as follows:

- To introduce DJ Trigonometry as a new and independent trigonometric framework based on square-oriented and polygonal geometry rather than circular geometry.
- To define and formalize the concepts of DJ Measure, DJ Degree, and DJ Angle as alternative units for angular and proportional measurement in closed geometric figures.
- To develop a specialized geometric instrument, the DJ Protractor, for accurately measuring DJ Angles and visualizing angular relationships within polygon-based geometries.
- To establish the DJ Badwan Theorem as a fundamental proportional relationship between the sides of a right-angled triangle within the DJ Trigonometric system, without reliance on classical trigonometric functions.
- To demonstrate that trigonometric relationships can be applied consistently to closed figures such as triangles, squares, and rectangles using proportional reasoning instead of circular angle functions.
- To simplify trigonometric learning by reducing conceptual dependence on radians, circular arcs, and trigonometric tables, thereby improving mathematical clarity and accessibility for learners.
- To provide a coherent theoretical foundation that complements classical trigonometry while offering an alternative geometric perspective for educational and analytical applications.

8. Discussion

The DJ Trigonometry framework developed in this study should not be interpreted as a replacement for classical trigonometry, but rather as a complementary and alternative geometric framework. The purpose of this discussion is to analyze the extent to which the concepts of DJ Measure, DJ Degree, and DJ Angle are capable of expressing angle-side relationships in closed and polygon-based geometric structures in a clear and consistent manner. Classical trigonometry is fundamentally grounded in circular geometry, where angles are defined through arcs, radii, and radian measures. This framework is highly effective for circular, conical, and spherical geometries. However, when the focus shifts to polygonal or closed linear figures, circular

interpretations may appear indirect or conceptually abstract. In this context, DJ Trigonometry offers a different perspective by establishing angle-side relationships through square-based geometry and proportional reasoning. The principal theoretical contribution of this research is the DJ Badwan Theorem. The discussion clarifies that this theorem is not a reformulation of the classical tangent function, but instead represents an independent proportional relationship defined entirely within the DJ measurement system. By expressing the relationship between the sides of a right-angled triangle through the DJ Measure, the framework demonstrates that certain trigonometric relationships can be formulated without reliance on circular trigonometric functions.

An important aspect of DJ Trigonometry is the deliberate definition of the DJ Measure as a bounded proportional quantity, with values strictly less than one. This constraint is not merely a technical limitation, but a necessary condition for preserving the triangular nature of the geometric configuration. As the DJ Measure approaches unity, the geometric behavior transitions toward square-like symmetry, at which point trigonometric interpretation becomes less meaningful. This observation highlights the internal consistency and definitional clarity of the DJ framework. The numerical use of 360 in DJ Degree, analogous to the classical degree system, also warrants discussion. This similarity is not intended to suggest mathematical equivalence, but rather to maintain pedagogical continuity. The geometric foundations of DJ Degree and classical degree are fundamentally different, and this study explicitly avoids direct comparability between the two systems. By addressing this distinction at the theoretical level, the framework minimizes potential conceptual ambiguity. The findings of this study suggest that DJ Trigonometry may be particularly effective for educational and conceptual applications, where clarity and proportional intuition are essential. At the same time, it is acknowledged that broader acceptance of this framework will require further comparative analysis, formal mathematical validation, and application-based investigations. Overall, this discussion establishes DJ Trigonometry as a structured, bounded, and conceptually consistent framework that coexists with classical trigonometry while offering an alternative perspective for the geometric analysis of polygon-based and closed figures.

9. Conclusion

This research presents DJ Trigonometry as a new and independent trigonometric framework aimed at simplifying the understanding of angular and proportional relationships in closed geometric figures. The primary objective of this study was to develop an alternative approach to classical trigonometry that reduces conceptual complexity while preserving mathematical consistency. This objective has been achieved by introducing the concepts of DJ Measure, DJ Degree, DJ Angle, and a dedicated geometric instrument, the DJ Protractor, all of which are based on square-oriented geometry rather than circular constructions.

Unlike traditional trigonometry, which relies heavily on circular angle measurement and trigonometric functions, DJ Trigonometry establishes direct proportional relationships between sides and angles. The DJ Badwan Theorem plays a central role in this framework by providing a straightforward method to determine side lengths in a right-angled triangle using proportional reasoning, without invoking sine, cosine, or tangent functions. This contributes significantly to improving conceptual clarity, especially at the foundational level of mathematical learning. Furthermore, the

proposed system demonstrates that angular relationships in closed and polygonal figures can be analyzed consistently without dependence on circular arcs, radians, or π . By adopting a square-based measurement structure, DJ Trigonometry extends trigonometric reasoning to geometric forms such as squares, rectangles, and general triangles in a unified manner. The outcomes of this research suggest that DJ Trigonometry has strong potential as an educational and pedagogical tool, offering an intuitive pathway for learners to understand geometric proportionality. At the same time, the framework provides a mathematically coherent alternative perspective that complements, rather than contradicts, classical trigonometry. In conclusion, DJ Trigonometry fulfills its stated objectives by presenting a simplified, proportional, and non-circular approach to trigonometric analysis. This work lays a conceptual foundation for further theoretical refinement, mathematical validation, and possible future applications in geometry education and applied mathematical modeling.

Compliance with Ethical Standards***Ethical Approval***

This research is purely theoretical and mathematical in nature. It does not involve any human participants, animals, clinical experiments, surveys, or personal data. Therefore, ethical approval from an institutional review board or ethics committee was not required.

Consent to Participate

Not applicable. This study does not involve human subjects.

Consent for Publication

Not applicable. No personal, clinical, or identifiable data of any individual has been used in this manuscript.

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Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this paper. The research was conducted independently and without any external influence.

Author Contributions

This manuscript is a single-author work. The author solely conceived the idea, developed the theoretical framework, performed the mathematical analysis, designed the DJ Trigonometric concepts, and wrote and revised the entire manuscript.

Data Availability Statement

All data analyzed during this study are included within the manuscript and are available from the corresponding author upon reasonable request.

Code Availability

Not applicable. No computational code or software was used or generated in this study.

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Volumetric Sensible-Enthalpy Formulation for Coupled Transport in MHD Casson Nanofluid Flow: Revealing Energy Pathways beyond Wall Fluxes

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Abstract

This study presents a comprehensive analysis of coupled momentum, heat, and mass transfer in MHD Casson nanofluid flow over a stretching surface using a volumetric sensible-enthalpy formulation. Unlike conventional temperature-based models, the approach explicitly resolves internal energy generation, storage, and redistribution arising from viscous dissipation, Joule heating, buoyancy forces, and nanoparticle transport. The governing nonlinear equations for momentum, enthalpy, and concentration are transformed via similarity variables and solved numerically using a collocation method. Numerical investigations show that magnetic fields strongly suppress fluid motion while enhancing internal energy accumulation, reducing surface heat and mass transfer. Thermal and solutal buoyancy synergistically intensify momentum, heat, and species transport. Viscous dissipation primarily contributes to volumetric energy storage rather than wall heat flux, whereas Brownian motion significantly redistributes internal enthalpy with negligible effect on skin friction. Increasing the Casson parameter enhances convective enthalpy transport by weakening yield-stress resistance. The results demonstrate that wall fluxes alone may not reliably indicate internal energy dynamics, highlighting the enthalpy-based framework as a unified tool for capturing coupled transport pathways in non-Newtonian MHD nanofluids.

Keywords: Boundary layer; viscous dissipation; Brownian motion; Magnetic field; stretching sheet; nonlinear, ODE

1. Introduction

Nanofluids, formed by dispersing nanoscale solid particles into conventional base fluids, have emerged as advanced heat-transfer media with enhanced thermal and transport characteristics compared to pure liquids [1]. Among these, Al_2O_3 ethylene glycol (EG) suspensions have received significant attention due to their superior thermal conductivity, stability, and compatibility with industrial cooling and energy systems [2]. The interaction between momentum, thermal, and solutal diffusion in such nanofluid flows plays a crucial role in determining heat and mass transport efficiency, especially in systems where magnetic fields, viscous dissipation, and Brownian motion are simultaneously active [3,4]. In electrically conducting nanofluids, the application of a transverse magnetic field induces a Lorentz force that resists motion, thereby influencing both hydrodynamic and thermal boundary layers. The study of magnetohydrodynamic (MHD) nanofluid flow has therefore become fundamental to modern technologies such as micro-electromechanical systems (MEMS), magnetically

controlled cooling, thermal energy storage, and biomedical coating flows. Furthermore, Casson rheology provides a robust model for non-Newtonian fluid structures relevant to polymeric suspensions, paints, and biofluids, capturing yield-stress effects that strongly couple with viscous and thermal fields [5, 6].

Aluminium oxide (Al_2O_3) nanoparticles are chemically stable, non-toxic, and cost-effective, which has made them among the most extensively studied nanomaterials. These nanoparticles possess high thermal conductivity and can be uniformly dispersed in a variety of base fluids, with ethylene glycol (EG) being particularly effective. Ethylene glycol itself is widely used as a base fluid due to its low volatility, moderate viscosity, and good compatibility with metal oxide nanoparticles. Its properties make it an ideal medium for suspending nanoparticles and it is commonly employed in automotive and electronic cooling applications. Nanofluids composed of Al_2O_3 ethylene glycol have demonstrated notable enhancements in thermal conductivity, specific heat capacity, and convective heat transfer performance, making them promising candidates for advanced cooling systems. For instance [7], conducted a numerical investigation on heat transfer enhancement using a water/ethylene glycol mixture containing $Al_2O_3 - TiO_2$ nanoparticles. [8], analyzed the thermal performance of Al_2O_3 -water nanofluid flow in composite microchannels. Similarly, [9] studied the heat transfer characteristics of an ethylene glycol-based Casson nanofluid around a horizontal circular cylinder under MHD effects, reporting that CuO nanofluid exhibited the largest thermal boundary layer thickness and the lowest Nusselt number. A method-of-moments analysis on $EG - Al_2O_3$ nanofluid flow through an expanding/contracting rectangular channel was conducted by [10].

Their results reveal that channel contraction intensifies velocity gradients and heat transfer, while expansion weakens the overall transport characteristics. The general trend from their study is that geometric deformation directly regulates nanofluid momentum and thermal diffusion. [11], developed and evaluated six supervised machine-learning models to predict the thermal conductivity of Al_2O_3 –ethylene glycol nanofluids. Their findings show that data-driven models can accurately estimate thermal conductivity based on nanoparticle concentration and fluid properties, with tree-based methods achieving the highest predictive accuracy. The general conclusion from their work is that machine-learning techniques offer a reliable alternative to empirical correlations for nanofluid thermal characterization. [12], performed detailed measurements of the thermal conductivity and viscosity of Al_2O_3 –ethylene glycol nanofluids across various temperatures and particle loadings. Their results show that adding alumina nanoparticles significantly increases thermal conductivity, with the enhancement becoming more pronounced at higher temperatures due to intensified Brownian motion. They also reported a substantial rise in viscosity with particle concentration, indicating a trade-off between improved heat transfer and increased flow resistance. The general implication of their study is that while EG-based alumina nanofluids offer superior thermal performance, their rheological penalties must be carefully considered in system design and pumping-power calculation. [13], conducted both experimental and numerical investigations into turbulent forced-convection heat transfer of Al_2O_3 –ethylene glycol nanofluids in a flat tube. They found that introducing Al_2O_3 nanoparticles into the ethylene glycol base fluid significantly increases the convective heat transfer coefficient, particularly at

higher Reynolds numbers, while numerical simulation validated the experimental trends across various nanoparticle volume fractions.

The general implication of their work is that Al_2O_3 –ethylene glycol nanofluids offer a viable route for heat-transfer augmentation in turbulent flow regimes, though accompanying increases in viscosity and potential frictional losses must be considered for practical implementation. [14], examined the turbulent forced convective flow of an Al_2O_3 –ethylene glycol nanofluid in a circular tube under constant wall-temperature conditions, with the added focus on entropy generation minimization through genetic-algorithm (GA) and direct optimization techniques. Their results show that convective heat transfer improves with increasing Reynolds number and nanoparticle volume concentration, but also that entropy generation rises with Reynolds number and smaller particle sizes thus revealing a trade-off between heat transfer enhancement and irreversibility. The general implication of their work is that optimal nanoparticle size and volume fraction exist which minimize entropy generation while maintaining improved thermal performance in EG-based alumina nanofluids. [15], carried out a numerical investigation of a hybrid nanofluid consisting of Al_2O_3 – Cu nanoparticles dispersed in ethylene glycol, modeled through a non-Newtonian Maxwell fluid formulation under MHD conditions over a stretching inclined sheet with suction in a porous medium. Their results demonstrate that the fluid velocity increases with the Maxwell fluid parameter, suction parameter and inclination angle, while both increasing the magnetic field strength and the porous medium parameter tend to reduce velocity but increase temperature. They found that the Nusselt number (heat-transfer rate) responds sensitively to suction and nanoparticle volume fractions, suggesting that the combination of MHD, non-Newtonian rheology and hybrid nanoparticle dispersion strongly influences thermal efficiency. The general implication of their study is that hybrid Al_2O_3 – Cu/EG nanofluids, when used under MHD and non-Newtonian frameworks, can achieve enhanced heat-transfer behaviour, but the interplay between magnetic, porous and rheological parameters must be carefully managed. [16], investigated the heat-transfer performance of Al_2O_3 , SiO_2 and ZnO nanoparticle suspensions in ethylene glycol–water ($EG - H_2O$) base fluids within a rectangular microchannel under laminar flow conditions. Their results show that increasing nanoparticle volume fraction and higher EG content in the base fluid led to enhanced convective heat transfer coefficients; for example, a 40% ($EG - H_2O$) mixture produced the highest increase with ZnO at 5% volume fraction. They also observed that pressure drop and pumping power rise significantly with EG ratio and nanoparticle loading, with ZnO showing the largest penalty.

The general implication is that while EG –based nanofluids with Al_2O_3 (and other oxides) can achieve considerable heat-transfer improvement in microchannel applications, system design must carefully balance enhancement against increased hydrodynamic losses. [17], carried out a numerical investigation of magnetohydrodynamic (MHD) boundary-layer flow of Al_2O_3 –ethylene glycol nanofluids (in an ethylene-glycol/water base) past a melting surface, introducing an effective Prandtl number model to account for nanoparticle effects. Their results demonstrate that inclusion of the effective Prandtl number leads to a thicker momentum and thermal boundary layer compared with conventional values, and that the transverse magnetic field suppresses fluid motion while enhancing thermal boundary-layer thickness. The general implication from their study is that in electrically

conductive nanofluids, both magnetic field and nanoparticle-modified transport properties (via effective Prandtl number) must be incorporated to accurately model melting heat-transfer processes. Similarly, [18] investigated the combined effects of Brownian motion, thermophoresis and magnetic field on a hybrid nanofluid flowing over a stretched wedge surface. Their results show that increasing the wedge angle enhances the local Nusselt number, whereas stronger magnetic field strength thickens the thermal boundary layer and reduces heat-transfer efficiency. The general implication of their work is that in hybrid nanofluid systems, the interaction of nanoparticle-diffusive mechanisms (Brownian, thermophoretic) with magnetic suppression and geometric parameters must be accounted for in heat-transfer models. [19], investigated the effects of Brownian motion and thermophoresis on nanofluid flow in channels using a two-phase model. Their results show that nanoparticle migration significantly alters both temperature and concentration fields, demonstrating that diffusive particle transport strongly enhances coupled heat-mass transfer. The general implication of their findings is that Brownian and thermophoretic forces must be included to accurately capture nanofluid behavior in confined geometries.

From the literature reviewed, it is evident that conventional nanofluid models predominantly employ temperature-based formulations of the energy equation and the thermal buoyancy. While these approaches have yielded valuable insights, they often overlook the fundamental volumetric enthalpy coupling that intrinsically links fluid density, heat capacity, and the temperature field. This oversight becomes particularly significant when considering the complex interplay between multiple transport mechanisms in Casson nanofluids under magnetic fields. To address this limitation, the present work introduces a volumetric sensible enthalpy formulation: $\hat{h} = \rho_{nf} c_{p,nf} (T - T_\infty)$, which directly embeds the nanofluids thermo-physical properties into the thermal buoyancy and the governing energy transport description. This formulation naturally captures the dynamic feedback between heat generation, viscous dissipation, Joule heating, and nanoparticle diffusion (Brownian motion), providing a more physically complete framework for multi-parameter volumetric sensible enthalpy transport analysis.

The primary objectives of this study are to analyze the influence of magnetic field intensity, viscous dissipation, buoyancy force, interacting transport parameters (Brownian motion, Schmidt number) and other thermophysical properties on the velocity, enthalpy, and concentration distributions, and quantify how these coupled effects alter skin-friction, Nusselt number, and Sherwood number characteristics within a volumetric enthalpy framework. While mathematically equivalent to temperature formulations for constant property cases, the volumetric enthalpy approach provides a more fundamental thermodynamic framework that directly represents the actual energy content per unit volume, offers enhanced physical insight into coupled transport mechanisms, and establishes a foundation for extending to more complex scenarios involving property variations, phase change, or large temperature ranges.

Most existing studies on MHD Casson nanofluid boundary-layer flow formulate thermal transport using temperature as the primary variable, thereby restricting analysis to wall heat-flux behaviour and external thermal boundary-layer characteristics. In contrast, the present work introduces a volumetric sensible-enthalpy formulation in which the energy equation and the thermal buoyancy are recast in terms of internal energy density, coupled with enthalpy-dependent thermal conductivity and

nanoparticle transport effects. This framework enables direct examination of internal energy storage, redistribution, and generation mechanisms within the nanofluid, providing physical insight that cannot be obtained from temperature-based models. The formulation reveals how Lorentz-force damping, Brownian diffusion, viscous dissipation, and dual buoyancy forces collectively regulate volumetric energy pathways rather than merely surface heat flux. Consequently, the study establishes magnetic-enthalpy coupling as a new control mechanism for internal thermal management in electrically conducting Casson nanofluids, offering a fundamentally different perspective on energy transport beyond conventional wall-flux-oriented analyses.

2. Mathematical Formulation of the Model

Consider a steady, laminar, two-dimensional boundary-layer flow of an electrically conducting Casson nanofluid consisting of Al_2O_3 nanoparticles dispersed in ethylene glycol (EG). The flow is induced by a continuously stretching sheet with a linear velocity $U_w = ax$, where $a > 0$ is the constant stretching rate. A uniform transverse magnetic field of strength B_0 is applied perpendicular to the surface. Nanoparticle transport is governed by Brownian motion, while thermal energy transport is described using a volumetric sensible enthalpy formulation defined by $\hat{h}(x, y) = \rho_{nf} c_{p,nf} (T - T_\infty)$, replacing the classical temperature variable in the thermal buoyancy and energy equation. Let $u(x, y)$ and $v(x, y)$ denote the velocity components in the streamwise (x) and transverse (y) directions, respectively. The dependent variables are the velocity field, the volumetric sensible enthalpy $\hat{h}(x, y)$, and the nanoparticle concentration $C(x, y)$ as displayed in Figure 1.

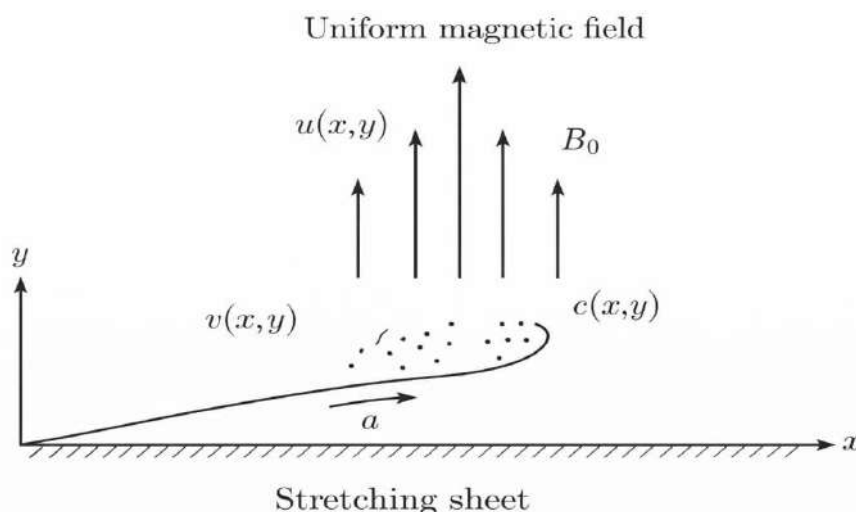


Figure 1. Schematic Diagram of the flow

The governing partial differential equations, satisfying boundary-layer assumptions are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_{nf}}{\rho_{nf}} \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} + g\beta_T \left(\frac{\hat{h}}{\rho_{nf} c_{p,nf}} \right) + g\beta_C (C - C_\infty) - \frac{\sigma_{nf} B_0^2 u}{\rho_{nf}} \quad (2)$$

$$\begin{aligned}
 u \frac{\partial \hat{h}}{\partial x} + v \frac{\partial \hat{h}}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{\kappa(\hat{h})}{\rho_{nf} c_{p,nf}} \frac{\partial \hat{h}}{\partial y} \right) + \tau D_B \frac{\partial C}{\partial y} \frac{1}{\rho_{nf} c_{p,nf}} \frac{\partial \hat{h}}{\partial y} + \frac{\mu_{nf}}{\rho_{nf} c_{p,nf}} \left(\frac{\partial u}{\partial y} \right)^2 \\
 &\quad + - \frac{\sigma_{nf} B_0^2 u^2}{\rho_{nf} c_{p,nf}} \quad (3) \\
 u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} &= D_B \frac{\partial^2 C}{\partial y^2} \quad (4)
 \end{aligned}$$

With the boundary conditions

$$\begin{aligned}
 u = ax, \quad v = 0, \quad -\kappa_f \frac{\partial \hat{h}}{\partial y} &= h_f (\hat{h}_f - \hat{h}), \quad C = C_w(x), \quad \text{at } y = 0 \\
 u \rightarrow 0, \quad \hat{h} \rightarrow \hat{h}_\infty, \quad C &\rightarrow C_\infty, \quad \text{at } y \rightarrow \infty \quad (5)
 \end{aligned}$$

Here u, v are the velocity components along x and y axis respectively, μ_{nf} and ρ_{nf} denote the dynamic viscosity and density of the nanofluid, ν is the kinematic viscosity, represents the variable thermal conductivity of the fluid as a function of enthalpy; β is the Casson fluid parameter, g denotes the gravitational acceleration; β_T and β_C are the thermal and solutal expansion coefficients, respectively. Furthermore, $\hat{h}$ is the dimensional volumetric enthalpy of the fluid (dimensional enthalpy), while $\hat{h}_\infty$ denotes the dimensional ambient (free stream) enthalpy far from the surface, which is taken as the reference enthalpy ($\hat{h}_\infty = 0$). The quantity $\hat{h}_w$ represents the dimensional wall (surface) enthalpy. The local temperature of the fluid, which is enthalpy-dependent, is denoted by T , while C represents the nanoparticle concentration. The symbols C_w and C_∞ denote the wall and free-stream concentrations, respectively. In addition, β_0 is the applied magnetic field strength, σ_{nf} denote the electrical conductivity of the nanofluid, $(\rho c_p)_{nf}$ is the effective heat capacity of the nanofluid, α_{nf} represent the thermal diffusivity of the nanofluid. The parameters D_B and D_T is Brownian diffusion and thermophoretic diffusion coefficient respectively. The quantity κ_f is the thermal conductivity of the base fluid, h denotes the dimensionless volumetric enthalpy corresponding to the fluid temperature, and $\hat{h}_f$ represent the volumetric enthalpy at fluid temperature, h_f is convective heat transfer coefficient of the base fluid, $U_w(x)$ is the stretching velocity along the plate, $C_w(x)$ is the concentration distribution along the plate, ν_{nf} and ν_{bf} denote the kinematic viscosities of the nanofluid and the base fluid, respectively.

The present model is formulated using the concept of volumetric sensible enthalpy of the nanofluid defined as

$$\hat{h} = \rho_{nf} c_{p,nf} (T - T_\infty), \quad (7)$$

from which the local temperature field is

$$T = T_\infty + \frac{\hat{h}}{\rho_{nf} c_{p,nf}}. \quad (8)$$

Where, ρ_{nf} is the effective density $c_{p,nf}$ the specific heat capacity at constant pressure, and T the local temperature expressed in terms of volumetric sensible enthalpy. This formulation establishes a direct correspondence between temperature and enthalpy,

allowing the temperature field to be entirely replaced by its enthalpy equivalent. Such a transformation simplifies the representation of internal volumetric effects, including viscous dissipation, Joule heating, magnetic field interactions, and Brownian motion of nanoparticles, within the boundary-layer framework. The incorporation of Brownian motion is essential for capturing nanoparticle-induced energy transport, which directly influences the local enthalpy and, consequently, the thermal behavior of the nanofluid. Following the approaches of [16, 20-22], the thermal conductivity of the flow is assumed to vary linearly with volumetric sensible enthalpy that is,

$$\kappa(\hat{h}) = \kappa^*(1 + \gamma h), \quad (9)$$

where κ^* represents the ambient thermal conductivity of the nanofluid, and h is the dimensionless enthalpy. In this study, the modified thermal conductivity variation parameter is defined as

$$\gamma = \delta \hat{h}_w, \quad (10)$$

such that δ is the thermal conductivity variation coefficient with units m^3/J , representing the rate of change of thermal conductivity with respect to the local volumetric enthalpy. Non-dimensionalization of the thermal diffusion coefficient, thermal diffusivity is defined as

$$\alpha_{nf} = \frac{\kappa^*}{\rho_{nf} c_{p,nf}}. \quad (11)$$

This parameter measures how rapidly heat diffuses through the nanofluid relative to its ability to store thermal energy, and it is the natural scaling quantity used when non-dimensionalizing thermal diffusivity in the enthalpy (energy) equation. The following properties for the nanofluid adopted [22-27] consist of effective density

$$(\rho_{nf}) = (1 - \phi)\rho_f + \phi\rho_p, \quad \text{dynamic viscosity } \frac{\mu_{nf}}{\mu_{nf} \nu_{bf}} = \frac{0.904e^{0.148\phi}}{1 - \phi + \phi \frac{\rho_{np}}{\rho_{bf}}}, \quad \text{specific heat:}$$

$$(\rho c_p)_{nf} = (1 - \phi)(\rho c_p)_f + \phi(\rho c_p)_p, \quad \text{thermal diffusivity } \alpha_{nf} = \frac{\kappa_{nf}}{\rho_{nf} c_{p,nf}}, \quad \text{and thermal}$$

$$\text{conductivity: } \frac{\kappa_{nf}}{\kappa_{bf}} = \frac{\kappa_{np} + 2\kappa_{bf} - 2\phi(\kappa_{bf} - \kappa_{np})}{\kappa_{np} + 2\kappa_{bf} + \phi(\kappa_{bf} - \kappa_{np})}.$$

Reducing the governing partial differential Equations (1)-(4) subject to the boundary conditions (5) and (6), the following similarity variables are introduced:

$$\eta = y \sqrt{\frac{a}{\nu_f}}, \quad \varphi(x, y) = \sqrt{a \nu_f} x f(\eta), \quad u = \frac{\partial \varphi}{\partial y}, \quad v = -\frac{\partial \varphi}{\partial x}, \quad (12)$$

$$\hat{h} = \hat{h}_w h(\eta) \quad (13)$$

$$C = C_\infty + (C_w - C_\infty)\phi(\eta). \quad (14)$$

The variable $h(\eta)$ which represents the dimensionless volumetric sensible enthalpy and satisfies the following boundary conditions is $h(\eta) = 0$ when $\hat{h}_\infty = 0$ (ambient condition, $T = T_\infty$), and $h(\eta) = 1$ when, $\hat{h} = \hat{h}_w$ (wall condition, $T = T_w$).

Substituting Equations (7)-(14) into Equations (1)-(4) reveals that Equation (1) is satisfied, while Equations (2)-(4), together with the boundary conditions (5)-(6), reduce to

$$\frac{\mu_{nf}}{v_f \rho_{nf}} \left(1 + \frac{1}{\beta} \right) f'''(\eta) + f(\eta) f''(\eta) - f'^2(\eta) + Gr_t + Gr_s - M f' = 0 \quad (15)$$

$$(1 + \gamma h) h''(\eta) + \gamma h'^2(\eta) + Pr_{nf} f(\eta) h'(\eta) + Nb \phi'(\eta) h'(\eta) + Br_h (f''(\eta))^2 + Pr_{nf} M Ec_h (f'(\eta))^2 = 0 \quad (16)$$

$$\phi'' + Sc f \phi' = 0. \quad (17)$$

Where prime denotes the differentiation with respect to η , $Gr_t = \frac{g \beta_T \hat{h}}{\rho_{nf} c_{p,nf} \alpha x^2}$, $Gr_s = \frac{g \beta_C (C_w - C_\infty)}{\alpha x^2}$, $Br_h = \frac{\mu_{nf} a^2 x^2}{\alpha_{nf} \rho_{nf} c_{p,nf} (\hat{h}_w)} \equiv \frac{\mu_{nf} U^2}{\kappa_{nf} (\hat{h}_w)}$, $Nb = \frac{\tau_{DB} (C_w - C_\infty)}{\kappa_{nf}}$, $Ec_h = \frac{a^2 x^2}{\rho_{nf} c_{p,nf} \hat{h}_w}$, $M = \frac{\sigma_{nf} B_0^2}{\rho_{nf} a}$, $Sc = \frac{v_f}{D_B}$ represents the enthalpy based thermal Grashof number, solutal Grashof number, volumetric Brinkman number, Brownian motion parameter, Eckert number, magnetic field parameter, and Schmidt number respectively. While $v_{nf} = \frac{\mu_{nf}}{\rho_{nf}}$ and $v_f = \frac{\mu_f}{\rho_f}$ denotes the kinematic viscosities of the nanofluid and the base fluid, subject to the dimensionless boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad h'(0) = -Bi(h(0) - \gamma), \quad \phi(0) = 1 \quad \text{at} \quad \eta = 0 \quad (18)$$

$$f'(\eta) \rightarrow 0, \quad h(\eta) \rightarrow 0, \quad \phi(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (19)$$

Employing these transformations, the coupled nonlinear ordinary differential Equations (15)–(17), with the dimensionless boundary conditions (18)–(19), are reduced to a system of linear ordinary differential equations. Let, $y_1(\eta) = f(\eta)$, $y_2(\eta) = f'(\eta)$, $y_3(\eta) = f''(\eta)$, $y_4(\eta) = h(\eta)$, $y_5(\eta) = h'(\eta)$, $y_6(\eta) = \phi(\eta)$, $y_7(\eta) = \phi'(\eta)$ such that

$$A = \frac{\mu_{nf}}{v_f \rho_{nf}} \left(1 + \frac{1}{\beta} \right) \quad (20)$$

$$\text{The corresponding first-order system associated with Equations (15)–(17) is given as} \\ y_1' = y_2 \quad (21)$$

$$y_2' = y_3 \quad (22)$$

$$y_3' = \frac{-y_1 y_3 + y_2^2 - Gr_t - Gr_s + M y_2}{A} \quad (23)$$

$$y_4' = y_5 \quad (24)$$

$$y_5' = \frac{-\gamma y_5^2 - Pr_{nf} y_1 y_5 - Nby_7 y_5 - Br_h y_3^2 - Pr_{nf} M Ec_h y_2^2}{1 + \gamma y_4} \quad (25)$$

$$y_6' = y_7 \quad (26)$$

$$y_7' = -Sc y_1 y_7, \quad (27)$$

subject to the reformulated boundary conditions

$$y_1(0) = 0, y_2(0) = 1, y_5(0) = -Bi(y_4(0) - \gamma), y_6(0) = 1, \\ y_1(0) = 0, y_2(0) = 1, y_3(0) = s_1, y_4(0) = s_2, y_5(0) = -Bi(s_2 - \gamma), y_6(0) = 1, \\ y_7(0) = s_3 \quad (29)$$

with initial conditions,

$$y_2(\infty) = 0, y_4(\infty) = 0, y_6(\infty) = 0$$

where, $s_1 = f''(0)$, $s_2 = h(0)$, and $s_3 = \phi'(0)$ represent the unknown initial slopes of the velocity, enthalpy, and concentration profiles, respectively. These parameters are iteratively adjusted until the far-field boundary conditions $y_2(\infty) = 0$, $y_4(\infty) = 0$, and $y_6(\infty) = 0$ are satisfied within a prescribed tolerance. The coupled nonlinear Equation (19) is solved using the collocation method implemented in SciPy's solve_bvp routine. This approach combines collocation with domain discretization to produce accurate and stable solutions for two-point boundary-value problems. Nonlinear differential systems arising in transport phenomena are commonly treated using collocation-based boundary-value solvers, while alternative numerical frameworks such as block linear multistep schemes for initial-value formulations have also been proposed in recent computational studies [28], reflecting continuous development of numerical strategies for complex transport models.

Using the similarity transformations in Equations (12)-(14), the skin-friction coefficient, Nusselt number, Sherwood number, and wall enthalpy are evaluated in terms of the similarity functions, and the corresponding wall flux relations; shear stress, sensible enthalpy flux, and mass flux are given as

$$\tau_w = \mu_{nf} \left(\frac{\partial u}{\partial y} \right)_{y=0}, q_w'' = -\kappa_w \left(\frac{\partial T}{\partial y} \right)_{y=0} = -\frac{\kappa_w}{\rho_{nf} c_{p,nf}} \left(\frac{\partial \hat{h}}{\partial y} \right)_{y=0}, j_w'' = -D_{nf} \left(\frac{\partial C}{\partial y} \right)_{y=0} \quad (31)$$

thermal conductivity at the wall, Skin friction coefficient, Nusselt number and Sherwood number is defined as

$$\kappa_w = \kappa_\infty [1 + \gamma h(0)], C_f = \frac{\tau_w}{\frac{1}{2} \rho_f U_w^2}, Nu = \frac{x q_w''}{\kappa_\infty \hat{h}_w}, Sh = \frac{x j_w''}{D_{nf} (C_w - C_\infty)} \quad (32)$$

Applying Equations (12)-(14) to (31), the dimensionless engineering quantities becomes

$$Cf Re^{1/2} = 2 \left(\frac{\mu_{nf}}{\mu_f} \right) f''(0), Nu Re^{-1/2} = -[1 + \gamma h(0)] h'(0), Sh Re^{-1/2} = -\phi(0) \quad (33)$$

3. Results and Discussion

A coupled transport in MHD Casson nanofluid flow under volumetric sensible-enthalpy formulation, revealing energy pathways beyond wall fluxes was considered. The governing partial differential Equations (2)-(4) subject to the boundary conditions (5)-(6) were reduced using similarity variables (12)-(13) to a coupled nonlinear ordinary differential Equations (15)-(17) with dimensionless boundary conditions in (18)-(19). The system of first order ODE was solved numerically using collocation method implemented in the solve_bvp routine of the SciPy library. The influence of the thermo-physical parametric analyses on the velocity, enthalpy, and concentration distributions is summarized in tabular form and illustrated graphically. The selected parameter values are consistent with those commonly adopted in the literature, and the corresponding baseline parameters are as follows:

$\beta = 0.2, Gr_t = 5.0, Gr_s = 2.0, M = 3.0, Pr_{nf} = 143, Nb = 0.1, Sc = 0.62, Ec_h = 1.0, Br_h = 0.01,$

Table 1 displays the base thermophysical properties of Al_2O_3 –ethylene glycol used in this study. The thermal conductivity and viscosity of ethylene glycol and Al_2O_3 –ethylene glycol nanofluids at temperatures spanning 283–323 K have been experimentally reported by [12]. Specific heat capacity data for Al_2O_3 and ethylene glycol nanofluids are based on measurements [29]. These sources provide values that support the approximate base properties used here.

Table 1: Base physical properties of Al_2O_3 –ethylene glycol nanofluid at 25°C

Property	Ethylene Glycol (Base fluid)	Al_2O_3 (Nanoparticles)
Density (ρ)	1110.0	3970.0
Specific capacity (c_p)	2415.0	765.0
Thermal conductivity (κ)	0.25	36.0

Table 2: C_f with Nb for different values of M

M	Nb	C_f
1.0	0.05	-0.5103288205063
1.0	0.1	-0.5102419762794
1.0	0.2	-0.5100678381552
1.0	0.3	-0.5098930976441
2.0	0.05	-0.6753711977589
2.0	0.1	-0.6752103403988
2.0	0.2	-0.6748878111238
2.0	0.3	-0.6745641909003
3.0	0.05	-0.8251779213817
3.0	0.1	-0.8249521861829
3.0	0.2	-0.8244995938047
3.0	0.3	-0.8240454990106
5.0	0.05	-1.0920738091469
5.0	0.1	-1.0917386306259
5.0	0.2	-1.091066662697
5.0	0.3	-1.0903925383055

Figure 2 illustrates the variation of the skin-friction coefficient $C_f Re^{\frac{1}{2}}$ with the Brownian motion parameter Nb for different values of the magnetic parameter M in an Al_2O_3 –ethylene glycol nanofluid. The skin-friction coefficient shows negligible sensitivity to variations in Nb (from 0.05 to 0.30) across all M , indicating that Brownian diffusion primarily influences the internal thermal and concentration fields without directly affecting wall shear stress. In contrast, increasing M from 1 to 5 causes a pronounced reduction in the skin-friction coefficient (more negative values), reflecting a significant increase in flow resistance at the wall due to the Lorentz force. This clear decoupling highlights that magnetic effects (M) dominantly control the hydrodynamic wall response, while Brownian effects (Nb) are confined to thermo-solutal transport within the fluid volume. This separation, discernible through the volumetric enthalpy formulation, exemplifies the core advantage of the present approach: it distinguishes mechanisms controlling wall-transfer rates from those governing internal energy and species pathways, a distinction often obscured in conventional temperature-gradient analyses.

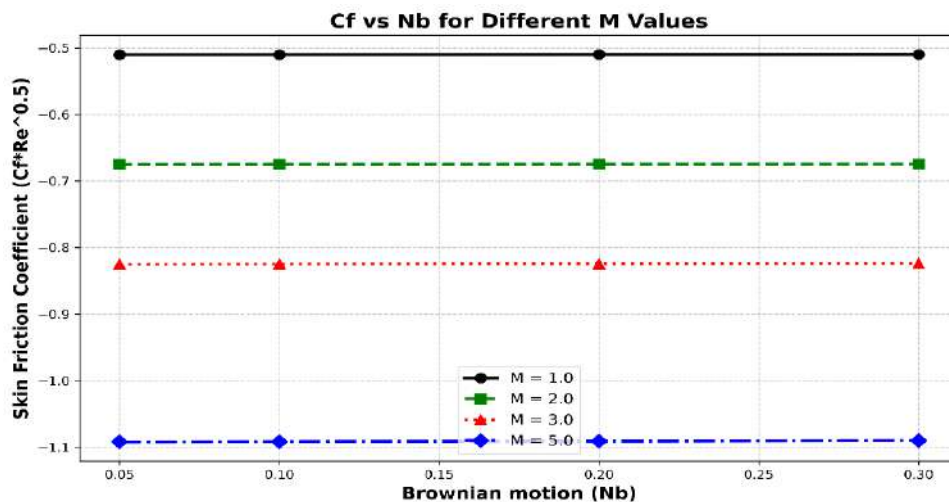


Figure 2. Variation of C_f with Nb for different M values

Table 3: Nu with Pr for different value of β

β	Pr_{nf}	Nu
0.1	100	-0.1576137920587
0.1	125	-0.1649765862556
0.1	150	-0.1710564442881
0.1	175	-0.176239613744
0.1	200	-0.1807598493767
0.2	100	-0.1459021576609
0.2	125	-0.1530585888260
0.2	150	-0.1589783298508
0.2	175	-0.1640317390021
0.2	200	-0.1684435251055
0.5	100	-0.1351651512912
0.5	125	-0.1421015583621
0.5	150	-0.147852391272
0.5	175	-0.1527678888243
0.5	200	-0.1570646243708
1.0	100	-0.130070522577

1.0	125	-0.1368863704084
1.0	150	-0.1425422671405
1.0	175	-0.1473822984193
1.0	200	-0.1516161506774

Figure 3 depicts the variation of the reduced Nusselt number $NuRe^{-1/2}$ with the nanofluid Prandtl number Pr_{nf} for different Casson parameter values β in the Al_2O_3 –ethylene glycol-based nanofluids. An increase in Pr_{nf} consistently reduces the Nusselt number for all β , indicating weakened wall heat transfer due to lower thermal diffusivity relative to momentum diffusivity. In the enthalpy-based formulation, higher Pr_{nf} suppresses internal energy diffusion across the thermal boundary layer, leading to a reduced wall enthalpy gradient and diminished conductive heat flux. For a fixed Pr_{nf} , increasing β significantly enhances the Nusselt number (Nu). Physically, larger β corresponds to reduced yield-stress effects, improving fluid mobility and momentum transport near the wall, which strengthens convective enthalpy transport and elevates heat transfer rates. These results highlight the competing roles of thermal diffusion governed by Pr_{nf} and rheological resistance controlled by β . The findings emphasize the importance of rheology–thermal coupling in Casson nanofluids and demonstrate that reducing yield-stress effects is crucial for maintaining efficient heat transfer in high Pr_{nf} regimes.

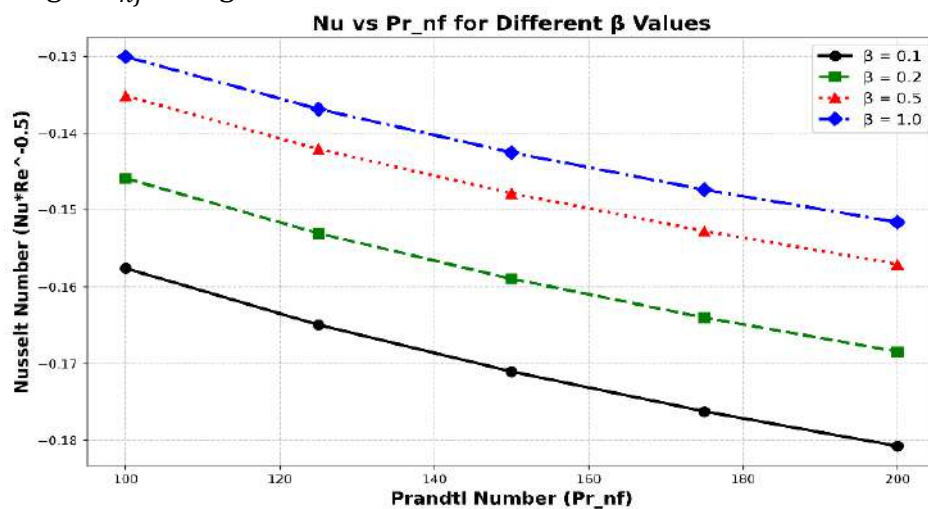


Figure 3. Variation of Nu with Pr_{nf} for different vales of β

Table 4: Effects of Sh and Sc for different vales of M

M	Sc	Sh
1.0	0.4	0.4340463927541
1.0	0.5	0.4897911837780
1.0	0.62	0.5504133669275
1.0	0.8	0.6320633351340
2.0	0.4	0.4145633061315
2.0	0.5	0.4696778295694
2.0	0.62	0.5297960100333
2.0	0.8	0.6109851220468
3.0	0.4	0.3976127324007
3.0	0.5	0.4521305896572
3.0	0.62	0.5117667537028

3.0	0.8	0.5925082009890
5.0	0.4	0.3692059036086
5.0	0.5	0.42250700098466
5.0	0.62	0.48113710391280
5.0	0.8	0.5609209644201

Figure 4 illustrates the variation of the reduced Sherwood number $Sh Re^{-1/2}$ with Schmidt number Sc for different magnetic parameter values of M in the present nanofluid. For all M , the Sherwood number increases monotonically with Sc , reflecting reduced mass diffusivity relative to momentum diffusivity and the consequent thinning of the concentration boundary layer. This leads to steeper wall concentration gradients and enhanced diffusive mass transfer. At a fixed Sc , increasing the magnetic parameter M markedly decreases the Sherwood number. The imposed transverse magnetic field induces a Lorentz force that retards fluid motion, weakens convective transport, and thickens the solutal boundary layer, thereby reducing the wall mass flux. This suppressive effect becomes more pronounced at higher Sc , where mass transfer is otherwise diffusion-dominated. Within the volumetric sensible-enthalpy framework, these trends demonstrate the competing influences of diffusion enhancement due to higher Sc and convective attenuation caused by magnetic damping. The results highlight the effectiveness of magnetic fields as a control mechanism for regulating nanoparticle mass transfer in nanofluid systems.

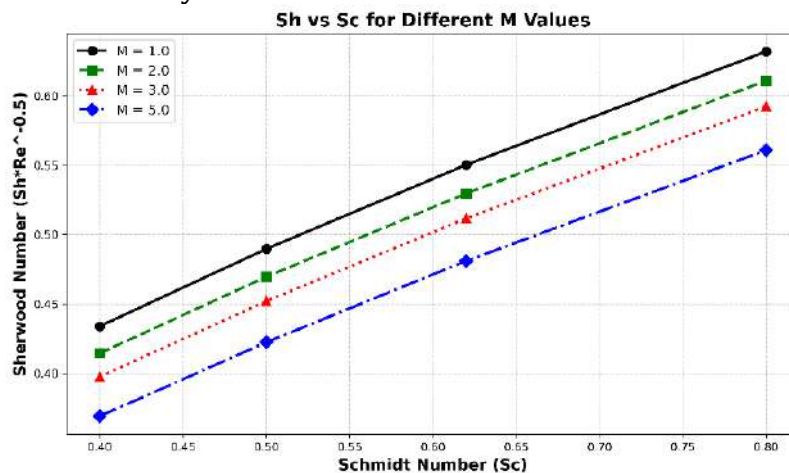


Figure 4. Variation of Sh and Sc for different values of M

Table 5: Numerical computational effects of C_f with Gr_t for different values of Gr_s

Gr_s	Gr_t	C_f
0.5	1.0	-1.5233462094851
0.5	3.0	-1.3826700984103
0.5	5.0	-1.234999526752
0.5	8.0	-0.9987092986578
0.5	10.0	-0.8300270155312
1.0	1.0	-1.3978139098906
1.0	3.0	-1.2496967197559
1.0	5.0	-1.0940878979074
1.0	8.0	-0.8449018411015
1.0	10.0	-0.6669312729140
2.0	1.0	-1.156676969882

2.0	3.0	-0.9949728397609
2.0	5.0	-0.8249521861829
2.0	8.0	-0.5525402389575
2.0	10.0	-0.3579863107303
5.0	1.0	-0.4861521204291
5.0	3.0	-0.2908907158815
5.0	5.0	-0.0856730411989
5.0	8.0	0.24262088297255
5.0	10.0	0.47646198323450

Figure 5 illustrates the variation of the scaled skin-friction coefficient $C_f Re^{1/2}$ with the thermal Grashof number Gr_t for various solutal Grashof numbers ($Gr_s = 0.5, 1.0, 2.0, 5.0$) for the considered fluid system. The coefficient increases monotonically with Gr_t for all Gr_s , reflecting enhanced fluid acceleration and wall shear stress due to stronger thermal buoyancy. For a fixed Gr_t , increasing Gr_s further augments $C_f Re^{1/2}$, demonstrating a synergistic effect between thermal and solutal buoyancy. At sufficiently high Gr_s , this combined buoyancy shifts the skin friction from negative to weakly positive values, indicating a change in the momentum boundary layer structure. Within the volumetric sensible-enthalpy framework, this trend results from the direct conversion of buoyancy-induced internal energy gradients into kinetic energy. The increase with both Gr_t and Gr_s shows how the formulation captures enhanced wall momentum transport through coupled thermal and solutal enthalpy. The synergy underscores the model's ability to unify dual buoyancy mechanisms thermal expansion and concentration-induced density variation into a single energy transport description, predicting combined flow acceleration and capturing regimes such as incipient flow reversal. These findings indicate that simultaneously increasing thermal and solutal buoyancy via greater temperature and concentration gradients can effectively enhance wall-driven momentum transport in buoyancy-assisted systems, with relevance for mixed-convection cooling and materials processing.

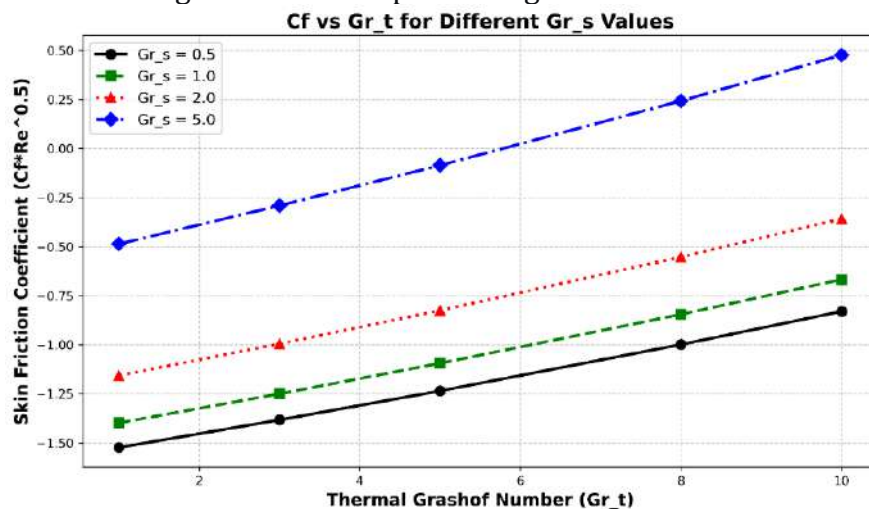


Figure 5. Variation of C_f and Gr_t for different values of Gr_s

Table 6: Tabulated values of Sh with Gr_s for different values of Gr_t

Gr_s	Gr_t	Sh
0.5	1.0	0.4682534056625
1.0	1.0	0.4799103498701

2.0	1.0	0.4997388521543
5.0	1.0	0.5440805187597
0.5	3.0	0.4737532637507
1.0	3.0	0.4855527928455
2.0	3.0	0.5056229805050
5.0	3.0	0.5504277545198
0.5	5.0	0.4795166524728
1.0	5.0	0.4914586405522
2.9	5.0	0.5117667537028
5.0	5.0	0.5570157913399
0.5	8.0	0.4887053651838
1.0	8.0	0.5008581046618
2.0	8.0	0.5215106909401
5.0	8.0	0.5673786784373
0.5	10.0	0.4952308514967
1.0	10.0	0.5075195473307
2.0	10.0	0.5283885365113
5.0	10.0	0.5746287000602

Figure 6 presents the variation of the reduced Sherwood number $Sh Re^{-1/2}$ with the solutal Grashof number Gr_s for different thermal Grashof numbers ($Gr_t = 1.0, 3.0, 5.0, 8.0$) for the considered fluid system. The Sherwood number increases monotonically with Gr_s for all Gr_t , indicating enhanced convective mass transfer due to stronger solutal buoyancy. For a fixed solutal Grashof number (Gr_s), increasing thermal Grashof number (Gr_t) further elevates the Sherwood number, demonstrating a synergistic interaction where thermal buoyancy augments the solutal-driven flow and intensifies the wall concentration gradient. Within the present model framework, this synergy arises from the coupled contribution of thermal and solutal effects to the total enthalpy, which drives convective transport. Increased thermal buoyancy converts more internal energy to kinetic energy, amplifying the velocity field responsible for species transport. This result highlights the model's capacity to predict coupled heat and mass transfer by unifying both buoyancy mechanisms in a single energy-based formulation. Practically, this synergy can be leveraged to optimize mass transfer in nanofluid applications, such as catalytic reactors or separation processes, through simultaneous control of thermal and solutal gradients.

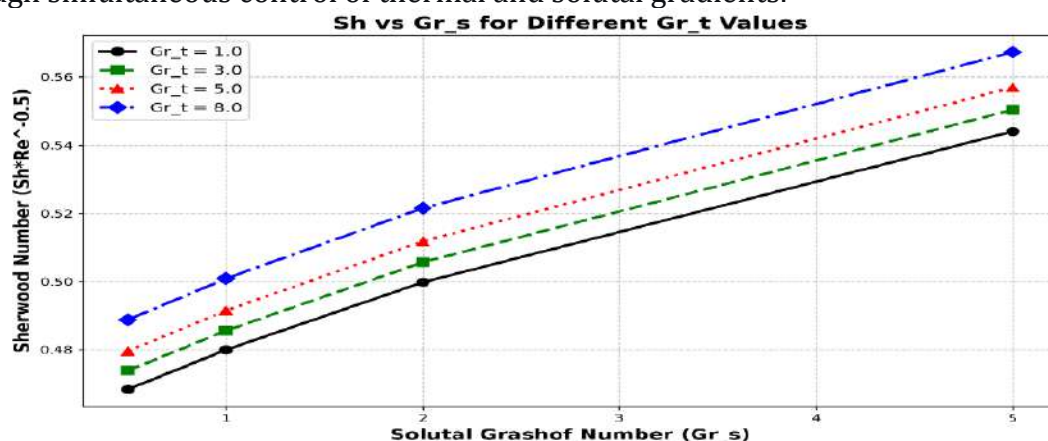


Figure 6. Variation of Sh with Gr_s for different Gr_t values

Table 7: Nu with Br_h for different Magnetic field values

M	Br_h	Nu
1.0	0.01	-0.0433656866383
1.0	0.05	-0.0433868570310
1.0	0.1	-0.0434133137013
1.0	0.5	-0.0436247144199
1.0	1.0	-0.0438883345829
2.0	0.01	-0.1017233269686
2.0	0.05	-0.1017562063612
2.0	0.1	-0.1017972962958
2.0	0.5	-0.1021256437599
2.9	1.0	-0.1025351491743
3.0	0.01	-0.1574205281773
3.0	0.05	-0.1574661661242
3.0	0.1	-0.1575232012783
3.0	0.5	-0.1579789915939
3.0	1.0	-0.1585475034886
5.0	0.01	-0.2639240529292
5.0	0.05	-0.2639976809173
5.0	0.1	-0.2640896977495
5.0	0.5	-0.2648251065934
5.0	1.0	-0.2657425545730

Figure 7 illustrates the variation of the reduced Nusselt number $Nu Re^{-1/2}$ with the Brinkman number Br_h for different magnetic parameter values ($M = 1.0, 2.0, 3.0, 5.0$). The results confirm that the Nusselt number exhibits negligible sensitivity to variations in Br_h across all magnetic field strengths, indicating that viscous dissipation contributes minimally to surface heat transfer within the examined parameter range. This behavior is explained within the volumetric sensible-enthalpy framework, where viscous heating primarily supplements the internal energy storage of the fluid rather than enhancing the wall-normal thermal gradient responsible for conductive heat flux. This clear partitioning—where viscous work contributes to volumetric energy storage rather than wall transfer—is precisely the kind of internal pathway revealed by the enthalpy formulation, moving beyond the assumption that all generated heat directly impacts surface flux. For a fixed Br_h , increasing the magnetic parameter M leads to a pronounced reduction in $Nu Re^{-1/2}$, reflecting the suppressive effect of the Lorentz force on fluid motion. Enhanced magnetic damping thickens the thermal boundary layer, weakens convective heat transport, and reduces the rate of heat transfer at the surface. The near-parallel nature of the curves further confirms that magnetic effects dominate over viscous dissipation in controlling surface heat transfer in the present MHD nanofluid system. From an engineering perspective, magnetic field regulation emerges as a more effective control strategy than viscous dissipation for tuning surface heat transfer in enthalpy-based MHD nanofluid thermal systems. This insight is particularly valuable for applications where active thermal management is required, such as in MHD-based heat exchangers or cooling devices, where adjusting the magnetic field strength offers precise control over heat transfer rates without relying on flow-induced viscous heating effects.

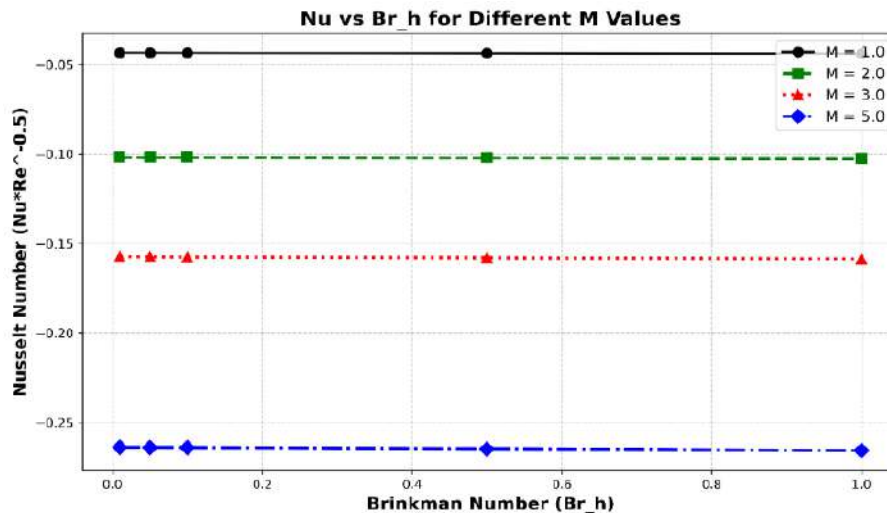


Figure 7. Variation of Nu vs Br_h for different M values

Table 8: Nu vs. Bi for Different Pr_{nf} values

Pr_{nf}	Bi	Nu
100	0.1	-0.0739633954530
100	0.2	-0.1459021576609
100	0.5	-0.3503951332875
100	1.0	-0.6577748358559
100	2.0	-1.1723645079990
125	0.1	-0.0774822357682
125	0.2	-0.1530585888260
125	0.5	-0.3690537056224
125	1.0	-0.6969747177599
125	2.0	-1.2547911422140
150	0.1	-0.0803951978620
150	0.2	-0.1589783298508
150	0.5	-0.3844689773184
150	1.0	-0.7293685537082
150	2.0	-1.3232083643823
175	0.1	-0.0828834056023
175	0.2	-0.1640317390021
175	0.5	-0.3976138827235
175	1.0	-0.7569890425817
175	2.0	-1.3817190286182

Figure 8 presents the variation of the reduced Nusselt number $Nu Re^{-1/2}$ with the Biot number Bi for different nanofluid Prandtl numbers ($Pr_{nf} = 100.0, 125.0, 150.0, 175.0$) of Al_2O_3 --ethylene glycol. The results demonstrate a clear monotonic decrease in $Nu Re^{-1/2}$ with increasing Bi across all Pr_{nf} values. This behavior indicates that enhanced surface convection (higher Bi) progressively reduces the wall heat transfer rate, as convective resistance at the boundary becomes more dominant relative to internal conductive heat transfer within the fluid. For a fixed Biot number, higher Pr_{nf} values consistently yield lower Nusselt numbers. Since $Pr_{nf} = \nu/\alpha_{nf}$ represents the ratio of momentum to thermal diffusivity, larger Pr_{nf}

values correspond to reduced thermal diffusion, which thickens the thermal boundary layer and suppresses conductive heat transfer from the wall. The parallel nature of the curves confirms that the influence of Bi is systematic across the examined range of thermal diffusivities, while the vertical separation between curves highlights the strong modulating effect of nanofluid thermal properties. Within the volumetric sensible-enthalpy framework, this interaction between Bi and Pr_{nf} reveals how boundary conditions and bulk fluid properties jointly govern heat transfer. The formulation captures the transition from conduction-dominated to convection-dominated regimes as Bi increases, while simultaneously accounting for the reduced thermal penetration depth associated with higher Pr_{nf} values. The engineering implication is significant: in nanofluid-based thermal systems, achieving optimal heat transfer requires careful balancing of surface convection conditions (characterized by Bi) and nanofluid thermal properties (characterized by Pr_{nf}). At high Bi and large Pr_{nf} , where heat transfer is most limited, design strategies should focus on either enhancing nanofluid thermal conductivity or adjusting surface convection characteristics to maintain adequate thermal performance.

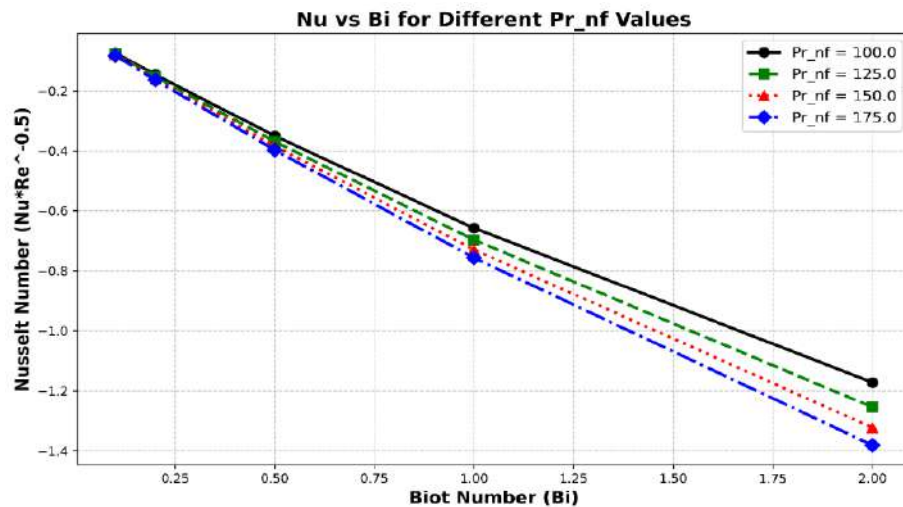


Figure 8. Variation of Nu with Bi for different Pr_{nf} values

Table 9. Nu and M for different Pr_{nf} values

M	Pr_{nf}	Nu
1.0	100	-0.0397529163974
2.0	100	-0.0942327167289
3.0	100	-0.1459021576609
5.0	100	-0.2439860596069
1.0	125	-0.0420011803138
2.0	125	-0.0988906846690
3.0	125	-0.1530585888260
5.0	125	-0.2563542341473
1.0	150	-0.0438519524315
2.0	150	-0.1027338119433
3.0	150	-0.1589783298508
5.0	150	-0.2666330934625
1.0	175	-0.0454256474777
2.0	175	-0.1060076874496

3.0	175	-0.1640317390030
5.0	175	-0.2754408845439
1.0	200	-0.0467949750203
2.0	200	-0.1088609110020
3.0	200	-0.1684435251055
5.0	200	-0.2831546405413

Figure 9 shows the reduced Nusselt number, $Nu Re^{-1/2}$, decreasing with the magnetic parameter M for different nanofluid Prandtl numbers ($Pr_{nf} = 100 - 175$) of Al_2O_3 --ethylene glycol. This indicates a consistent suppression of convective heat transfer by the magnetic field, attributable to the Lorentz force retarding flow and thickening the thermal boundary layer. For a fixed M , higher Pr_{nf} yields a lower Nusselt number. For the studied Al_2O_3 --ethylene glycol nanofluid, increased Pr_{nf} signifies a dominant rise in effective viscosity, which amplifies the Lorentz damping. The parallel curves confirm that magnetic suppression operates uniformly across this range. The results, framed within a volumetric sensible-enthalpy formulation, establish M as a key control parameter, where magnetic effects dominate thermal diffusion at higher values. This offers a direct mechanism for regulating heat transfer in MHD thermal systems. The greater suppression at higher Pr_{nf} further suggests that thermal performance can be tuned through nanoparticle loading. The distinct roles of magnetic damping (convective suppression) and Prandtl number (diffusive resistance) revealed here underscore the formulation's strength: it isolates how different physical mechanisms one external (Lorentz force) and one intrinsic (fluid properties)—separately modulate enthalpy transport from the volume to the wall.

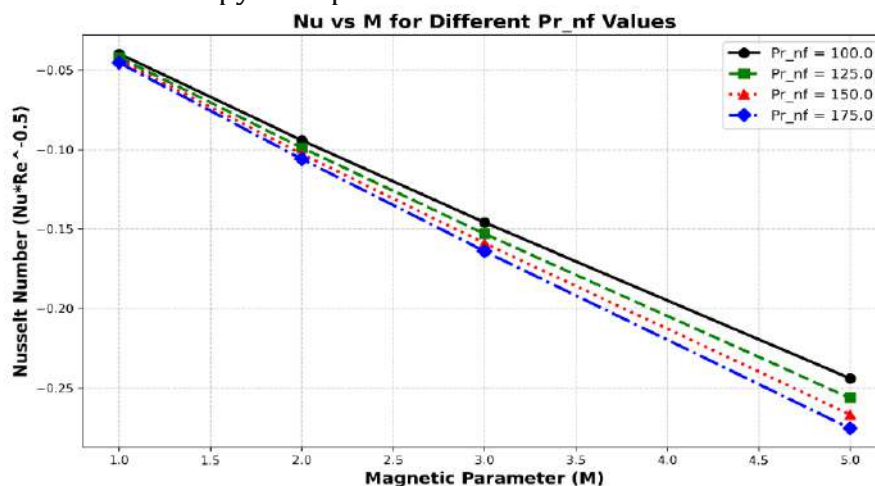


Figure 9. Variation of Nu with M for different values of Pr_{nf}

Table 10: Tabulated values of Sh with M for different values of Sc

Sc	M	Sh
0.4	1.0	0.434046393
0.4	2.0	0.414563306
0.4	3.0	0.397612732
0.4	5.0	0.369205904
0.5	1.0	0.489791184
0.5	2.0	0.46967783
0.5	3.0	0.45213059

0.5	5.0	0.422507001
0.62	1.0	0.550413367
0.62	2.0	0.52979601
0.62	3.0	0.511766754
0.62	5.0	0.481137104
0.8	1.0	0.632063335
0.8	2.0	0.610985122
0.8	3.0	0.592508201
0.8	5.0	0.560920964

Figure 10 presents the variation of the reduced Sherwood number $Sh Re^{-1/2}$ with the magnetic parameter M for different Schmidt numbers Sc in the present Al_2O_3 -ethylene glycol nanofluid. The Sc values are employed as dimensionless parameters to explore the momentum-to-mass-diffusivity ratio. A monotonic decrease in $Sh Re^{-1/2}$ with increasing M is observed for all Sc , indicating suppression of wall mass transfer by the magnetic field. This is attributed to the Lorentz force, which opposes fluid motion, weakens convective transport, and thickens the concentration boundary layer, thereby reducing the wall concentration gradient. For a fixed M , higher Schmidt numbers yield larger Sherwood numbers. Since $Sc = \nu/D$, an increase in Sc corresponds to lower mass diffusivity, which sharpens the near-wall concentration profile and enhances diffusive flux. This effect partially offsets the magnetic suppression of convection, revealing competition between magnetic damping and diffusive mass transfer. Within the volumetric sensible-enthalpy framework, these trends highlight the interplay between electromagnetic damping and species diffusion. The formulation captures how magnetic fields attenuate the convective component, while the Schmidt number governs diffusive resistance, a dual control mechanism crucial for systems requiring precise species transport regulation. From an engineering perspective, magnetic field strength serves as an active control parameter for modulating mass transfer in applications such as MHD-based separations, electrochemical reactors, and coating processes. Furthermore, by tailoring the effective Schmidt number through nanoparticle characteristics (size, concentration) or base-fluid properties, mass transfer performance can be fine-tuned under varying magnetic conditions. This quantified competition, resolvable within the volumetric enthalpy framework, exemplifies its utility: it disentangles the magnetic suppression of convection from the diffusive resistance governed by the Schmidt number, a coupling often conflated in models that consider only surface fluxes.

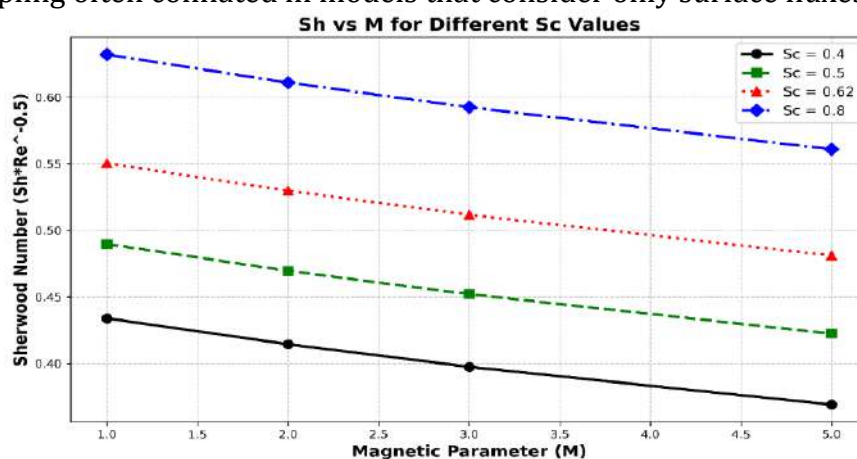


Figure 10. Variation of Sh with M for different Sc values

4. Conclusion

This study investigated magnetohydrodynamic Casson nanofluid flow and coupled heat-mass transfer over a stretching sheet using a volumetric sensible-enthalpy formulation. The effects of Brownian motion, magnetic field strength, Prandtl and Schmidt numbers, buoyancy parameters, viscous dissipation, Biot number, and rheological properties on skin friction, Nusselt number, and Sherwood number were systematically examined. The results revealed that:

1. The magnetic parameter exerts dominant control over momentum, heat, and mass transfer through Lorentz-force-induced damping.
2. Brownian motion has negligible influence on wall shear stress but plays a role in internal thermo-solutal transport.
3. Increasing Prandtl number suppresses wall heat transfer by reducing thermal diffusivity
4. Larger Casson parameter values enhance convective transport by weakening yield-stress resistance.
5. Schmidt number variations showed that reduced mass diffusivity sharpens concentration gradients and strengthens diffusive mass flux, partially offsetting magnetic suppression of species transport
6. Dual thermal and solutal buoyancy effects were found to synergistically accelerate flow and enhance wall momentum and mass transfer.
7. Viscous dissipation contributed minimally to surface heat transfer compared with magnetic damping.
8. Biot number governed surface convection resistance, while magnetic-Prandtl coupling demonstrated that Lorentz forces provide an effective means of regulating thermal transport in nanofluids.

These trends are consistent with classical MHD nanofluid studies in temperature-based formulations [30-35], although the present volumetric enthalpy approach provided additional insight into internal energy storage pathways, and successfully captured the interplay between electromagnetic damping, rheological behaviour, and thermo-solutal diffusion, providing a unified description of energy, momentum, and species transport in Casson nanofluid stretching-sheet systems. Similarly, this paper establishes that magnetic field strength is the most influential parameter controlling wall shear stress, heat transfer, and mass transfer in Casson nanofluid flow over a stretching sheet. Lorentz-force-induced damping thickens momentum, thermal, and concentration boundary layers, thereby suppressing convective transport at the surface. In contrast, Brownian motion primarily affects internal energy and concentration redistribution without significantly modifying wall momentum transport.

Thermal transport governed jointly by Prandtl number and Casson rheology: reduced thermal diffusivity weakens wall heat transfer, whereas diminished yield-stress effects enhance convective enthalpy transport. Mass transfer characteristics are dictated by Schmidt number and magnetic damping, with diffusion-driven enhancement competing against magnetic suppression of convection. Dual thermal and solutal buoyancy effects synergistically reinforce momentum and mass transport, confirming the importance of coupled buoyancy mechanisms in mixed-convection stretching-sheet flows. The volumetric sensible-enthalpy framework proved effective in unifying these coupled physical mechanisms, enabling accurate prediction of hydrodynamic, thermal, and species transport behaviours in complex MHD Casson nanofluid systems. The

results demonstrate that heat and mass transfer rates in Casson nanofluids can be actively regulated through magnetic field control and passively tuned through nanofluid thermophysical and rheological properties.

Our findings in this study are directly relevant to industrial and technological processes involving stretching surfaces where controlled heat and mass transfer are essential. Key application areas include:

1. Polymer extrusion and sheet stretching processes, where uniform cooling and surface quality depend on precise control of convective heat transfer.
2. Continuous casting and glass fiber drawing, where stretching-sheet flows govern solidification and material properties.
3. Coating and thin-film deposition technologies, where regulated mass transfer determines coating thickness and uniformity.
4. Textile and paper drying systems, where stretching surfaces and controlled heat fluxes improve energy efficiency.
5. MHD-assisted manufacturing systems, where magnetic fields are employed to regulate flow stability and thermal transport.
6. Micro-fabrication and roll-to-roll processing, where nanofluid-based cooling over moving sheets enhances thermal management.

The ability to manipulate momentum, heat, and mass transfer through magnetic fields and nanofluid design makes the present enthalpy-based Casson nanofluid stretching-sheet model valuable for next-generation materials processing and thermal management technologies.

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Nomenclature

h	Dimensionless enthalpy	$h = \frac{\hat{h} - \hat{h}_\infty}{\hat{h}_w - \hat{h}_\infty} = \frac{\hat{h}}{\hat{h}_w} = \frac{T - T_\infty}{T_w - T_\infty} = \theta$
$\hat{h}$	Dimensional volumetric sensible enthalpy	$\hat{h} = \rho_{nf} c_{p,nf} (T - T_\infty)$
$\hat{h}_w$	Wall stream volumetric sensible enthalpy	$\hat{h}_w = \rho_{nf} c_{p,nf} (T_w - T_\infty)$

$$\begin{array}{ll} \hat{h}_{\infty} & \text{Free stream volumetric sensible enthalpy } \hat{h}_{\infty} = 0 \\ T & \text{Local temperature field } T = T_{\infty} + \frac{\hat{h}}{\rho_{nf} c_{p,nf}} \end{array}$$

Author Contribution Statement: Onojovwo Felix Tega: Conceptualization, Analysis, Software, Writing-Original Draft, Oyem Onyekachukwu Anselm: Supervision; Validation, Review and Editing

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Iwuamadi's Primality Test

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Abstract

The formulation of a fast, one-time, and deterministic primality test is the motivation behind this research paper. The main mathematical formulae were researched to achieve the objectives mentioned above. This paper covers theoretical use, while a sister paper would spell the practical applications—theoretical experience is pro rata to practical experience (reality).

Keywords: Iwuamadi, Primality, Testing.

1. Introduction

Iwuamadi's primality test is a one-time, deterministic test with ease in identifying primes. The main formulae are from the triangular array below.

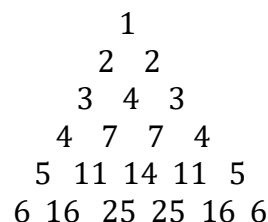
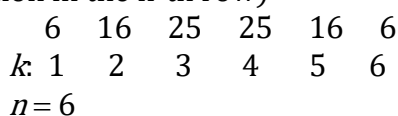


Figure 1

n = row ; k = column (position in the n -th row)



$$T(n, k) = C(n, k-1) + C(n-1, k) \quad (1)$$

$T(n, k)$ = entry

To test whether n is a prime, $n > 2$ and $2 \leq k \leq n-1$ ($n \in \mathbb{Z}^+$, $k \in \mathbb{Z}^+$)

In an n -th row, entries ($k=2$ to $k=n-1$) begin by subtracting 1 from entry $k=2$ followed by adding 1 to entry $k=3$, and alternating others in that order. And if the result of every entry is divisible by the number n , definitely, n is a prime, and if not divisible by number n , n is a composite.

$$R(n, k) = C(n, k-1) + C(n-1, k) - (-1)^k \quad (2)$$

$$R(n, k) = \text{entry} - (-1)^k \quad (3)$$

$R(n, k)$ = result (divided by n)

2. Preliminaries

Lemma 2.1:

$$R(n, k) = C(n, k-1) + C(n-1, k) - (-1)^k \quad (4) \quad (\text{derived from figure 1})$$

1)

Proof

$T(n, k)$ = entry in the triangular array of figure 1

$R(n, k)$ = result by subtraction/addition of 1 from/to $T(n, k)$, (which begins with subtraction and $k=2$)

k = divisor of n

n = tested number (the n -th row)

Figure 2 below shows $R(n, k)$ entries divided by n (of n -th rows) from $k=2$ to $k=n-1$, where n is the tested number for primality, and $n \geq 3$.

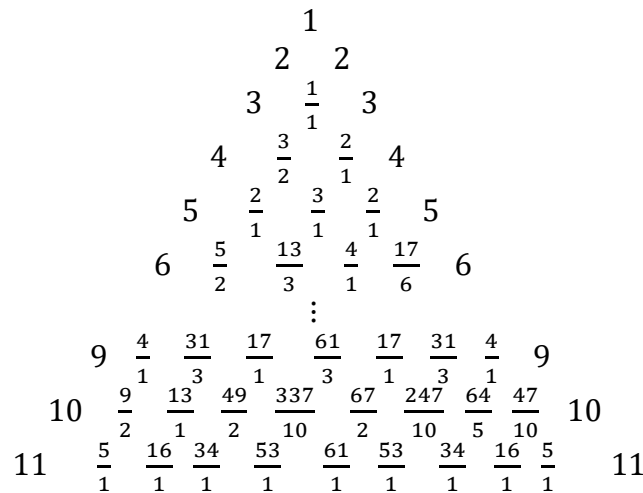


Figure 2

When $n=3$,

for $k=2$ (the only entry when $n=3$),

$$\begin{aligned} \frac{4-1}{n} &= \frac{3}{3} \\ &= \frac{1}{1} \quad (k=2 \text{ is not a factor of } n=3) \end{aligned}$$

$k=2$ is not a factor of $n=3$, and therefore, the denominator 1 is not a multiple of $k=2$.

When $n=4$,

for $k=2$,

$$\begin{aligned} \frac{7-1}{n} &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

$k=2$ is a factor of $n=4$, and therefore, the denominator 2 is a factor of $k=2$;

for $k=3$,

$$\begin{aligned} \frac{7+1}{n} &= \frac{8}{4} \\ &= \frac{2}{1} \end{aligned}$$

$k=3$ is not a factor of $n=4$, and therefore, the denominator 1 is not a multiple of $k=3$.

When $n=5$,

for $k=2$

$$\begin{aligned} \frac{11-1}{n} &= \frac{10}{5} \\ &= \frac{2}{1} \end{aligned}$$

$k=2$ is not a factor of $n=5$, and therefore, the denominator 1 is not a multiple of $k=2$;

for $k=3$,

$$\frac{14+1}{n} = \frac{15}{5}$$

$$= \frac{3}{1}$$

$k=3$ is not a factor of $n=5$, and therefore, the denominator 1 is not a multiple of $k=3$; for $k=4$,

$$\frac{11-1}{n} = \frac{10}{5} = \frac{2}{1}$$

$k=4$ is not a factor of $n=5$, and therefore, the denominator 1 is not a multiple of $k=4$.

Observe the trend with $n=6, 9, 10, 11$, you will deduce that when n is a composite, where k is a factor, the denominator is always the multiple of k , and where k is not a factor, the denominator is always a nonmultiple of k , and when n is a prime, the denominators are always 1 (a nonmultiple of k).

Therefore, $R(n, k)$ fits the primality test in the triangular array of figure 1. ■

Lemma 2.2:

$$R(n, \frac{n}{2}) = C(n, \frac{n}{2} - 1) + C(n - 1, \frac{n}{2}) - (-1)^{\frac{n}{2}} \quad (5) \quad (\text{middle entries})$$

when

$$n \text{ is even) } R_{me}(n, k) =$$

$$R(n, \frac{n}{2})$$

Proof

When $k = \frac{n}{2}$, where n is even, means, k is a factor of n , and therefore, the denominator will be a multiple of k . Remember, $2 \leq k \leq n - 1$.

Therefore, $R(n, \frac{n}{2})$ fits the primality test where n always comes as composite. ■

Lemma 2.3:

$$R(n, \frac{n+1}{2}) = C(n, \frac{n+1}{2} - 1) + C(n - 1, \frac{n+1}{2}) - (-1)^{\frac{n+1}{2}} \quad (6) \quad (\text{middle entry when } n \text{ is odd})$$

$$R_{mo}(n, k) = R(n, \frac{n+1}{2})$$

Proof

$$R(n, k) = T(n, k) - (-1)^k$$

Using $T(n, k)$,

$$\begin{aligned} T(n, k) &= C(n, k-1) + C(n-1, k) \\ &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{(n-1)!}{(n-1-k)!k!} \\ &= \frac{n!k!(n-k-1)!(n-1)!(k-1)!(n-k+1)!}{(n-k+1)!(k-1)!(n-k-1)!k!} \\ &= \frac{(n-1)!(k-1)!(n-k-1)!(nk+(n-k+1)(n-k))}{(n-k+1)!(k-1)!(n-k-1)!k!} \\ &= \frac{(n-1)!(nk+n^2-nk-nk+k^2+n-k)}{(n-k+1)!k!} \\ &= \frac{(n-1)!(n^2+k^2-nk+n-k)}{(n-k+1)!k!} \\ &= \frac{(n-1)!(n(n-k+1)+k(k-1))}{(n-k+1)!k!} \end{aligned}$$

When $k = \frac{n+1}{2}$,

$$\begin{aligned}
 &= \frac{(n-1)!(n^2 - n(k-1) + k(k-1))}{(n-k+1)!k!} \\
 &= \frac{(n-1)!(n^2 + (k-1)(k-n))}{(n-k+1)!k!} \\
 &= \frac{(n-1)!(n^2 + (\frac{n+1}{2}-1)(\frac{n+1}{2}-n))}{(n-\frac{n+1}{2}+1)!(\frac{n+1}{2})!} \\
 &= \frac{(n-1)!(n^2 + (\frac{n-1}{2})(\frac{1-n}{2}))}{(\frac{n+1}{2})!(\frac{n+1}{2})!} \\
 &= \frac{(n-1)!(n^2 - (\frac{n-1}{2})^2)}{((\frac{n+1}{2})!)^2} \\
 &= \frac{(n-1)!(\frac{n+1}{2})(\frac{3n-1}{2})}{((\frac{n+1}{2})!)^2}
 \end{aligned}$$

$\frac{3n-1}{2}$ is a multiplier that forms a linear sequence for n .

$\frac{(n-1)!(\frac{n+1}{2})}{((\frac{n+1}{2})!)^2}$ is the coefficient of $\frac{3n-1}{2}$ with distinct character, which differentiates primes from composites. $\frac{(n-1)!(\frac{n+1}{2})}{((\frac{n+1}{2})!)^2}$ separately identifies primes (named Iwuamadi's second primality test) without requiring $\frac{3n-1}{2}$.

$$\frac{(n-1)!(\frac{n+1}{2})}{((\frac{n+1}{2})!)^2} + 2(-1)^{\frac{n+1}{2}} \bmod n = 0 \quad (\text{Iwuamadi's second primality test; } n \text{ is prime, and prime only, otherwise, } n \text{ is composite.})$$

Arrangements and values from $\frac{(n-1)!(\frac{n+1}{2})}{((\frac{n+1}{2})!)^2}$:

For $n = 3$: $\frac{2!}{1!2!} = \frac{2}{2 \times 1}$

For $n = 5$: $\frac{4!}{2!3!} = \frac{3 \times 4}{3 \times 2 \times 1}$

For $n = 7$:
$$\frac{6!}{3!4!} = \frac{4 \times 5 \times 6}{4 \times 3 \times 2 \times 1}$$

For $n = 9$
$$\frac{8!}{4!5!} = \frac{5 \times 6 \times 7 \times 8}{5 \times 4 \times 3 \times 2 \times 1}$$

For $n = 11$
$$\frac{10!}{5!6!} = \frac{6 \times 7 \times 8 \times 9 \times 10}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

For $n = 13$
$$\frac{12!}{6!7!} = \frac{7 \times 8 \times 9 \times 10 \times 11 \times 12}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

...

For $n = 25$
$$\frac{24!}{12!13!} = \frac{13 \times 14 \times 15 \times 16 \times 17 \times 18 \times 19 \times 20 \times 21 \times 22 \times 23 \times 24}{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

Apply the dividend-divisor pair to the arrangements above beginning from the far right. It would be observed that when n is a composite, in at least a pair (apart from divisor = 1), divisor of a pair is a factor of its corresponding dividend, and when n is a prime, no pair has a divisor as a factor apart from divisor = 1. Divisibility in dividend-divisor pairs associated with composites makes it impossible for composites to behave as primes in $R\left(n, \frac{n+1}{2}\right)$. ■

3. Main results

$$R\left(n, \frac{n}{2}\right) \equiv \emptyset \pmod{n} \quad (n \text{ is a composite}) \quad (7)$$

When:

$$R\left(n, \frac{n+1}{2}\right) \equiv 0 \pmod{n} \quad (n \text{ is a prime}) \quad (8)$$

$$R\left(n, \frac{n+1}{2}\right) \equiv \emptyset \pmod{n} \quad (n \text{ is a composite}) \quad (9)$$

4. Applications

1. $R\left(n, \frac{n}{2}\right) \pmod{n}$ (one-time trial)

$$R\left(n, \frac{n}{2}\right) \pmod{n} \neq 0 \quad (n \text{ is a composite}) \quad (10)$$

2. $R\left(n, \frac{n+1}{2}\right) \pmod{n}$ (one-time trial)

$$R\left(n, \frac{n+1}{2}\right) \pmod{n} = 0 \quad (n \text{ is a prime}) \quad (11)$$

$$R\left(n, \frac{n+1}{2}\right) \pmod{n} \neq 0 \quad (n \text{ is a composite}) \quad (12)$$

5. Conclusion

Iwuamadi's primality test has been theoretically tested, and awaits its speed (operating time).

Reference

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The Impact of Waning Immunity on Covid-19 Dynamics: A Sqit Model with Contact Tracing and Quarantine

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Abstract

Waning immunity is a major driver of COVID-19 reinfection, yet most SQIT models assume permanent immunity in treated individuals. This study develops an SQIT model with contact tracing, quarantine, and a waning rate ϕ from treated class T back to susceptible class S . The model is formulated as a system of nonlinear ordinary differential equations. The disease-free equilibrium and basic reproduction number R_0 are derived using the next generation matrix method. Stability analysis shows the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$. Numerical simulations reveal that increasing ϕ accelerates the return treated individuals to the susceptible class, causing higher infection peaks and sustained transmission despite contact tracing and quarantine. Sensitivity analysis identifies ϕ , contact tracing rate, and house hold quarantine can control transmission only when ϕ is below a critical threshold. The study concludes that waning immunity must be incorporated in control strategies to avoid overestimating the impact of contact tracing and quarantine on COVID-19.

Keywords: COVID-19, SQIT model, Waning immunity, Contact tracing, Quarantine.

1. Introduction

COVID-19 is a respiratory disease caused by SARS-CoV-2, first identified in Wuhan, China, in late 2019. Transmission occurs mainly through respiratory droplets and aerosols from infected individuals, with contact transmission possible via contaminated surfaces. These droplets can land in the mouths or noses of people who are nearby or possibly be inhaled into the lungs. Spread is more likely when people are in close contact with one another [1]. The best way to prevent the transmission of infection is to avoid or limit contact with people who are showing symptoms of COVID-19 or respiratory infections. The next best thing is to practice good hygiene and physical distancing to prevent bacteria and viruses from being transmitted. Other prevention tips are to, wash hands frequently for at least 20 seconds at a time with warm water and soap, sanitizing of hands with an alcohol-based sanitizer, refuse touching one's face, eyes, nose, or mouth when hands are dirty, Staying at least 6 feet (2 meters) away from people, avoiding crowd gathering, covering of mouth with a tissue or the inside of one's elbow whenever there is sneeze or cough, disposing of used tissues immediately, and when at public places where it is difficult to maintain physical distancing, wear a face mask that cover both mouth and nose [1]. Contact tracing and quarantine reduced transmission when implemented promptly. In the UK, modeling estimated combined isolation and tracing reduced transmission by 50-60%, compared to 2-30% for self-isolation alone. A 7-day quarantine with a lateral flow test averted -50% of transmission, while 14-day quarantine without testing averted -59%. [2] found that

with $R_0 = 2.5$, control was probable if more than 50% of contacts were traced and isolated quickly.

Regional performance varied. In the WHO South-East Asia Region, contact tracing performed well when resourced and caseloads were low, but utility declined as incidence increased. As of 1 Feb 2023, WHO reported 10,120,175 cumulative cases and 1,472,174 deaths in the south-East Asia Region, and 757,599,137 cases and 6,843,908 death globally [4]. Africa reported lower infection rates at 0.011 but the highest cumulative case fatality rate at 0.020, compared to Europe's 0.326 infection rate and 0.009 fatality rate [4-5]. Mathematical studies confirm the effect. [7] estimated that intensive contact tracing, quarantine, and isolation in China could reduce the control reproduction number substantially, with travel restrictions cutting infections in Beijing by 91% over 7 days. [8] showed that tracing 40% or more of contacts flattened epidemic curves. [9] used an age-structured SEIR model for Canada and found that contact tracing and quarantine of exposed individuals reduced peak hospitalizations by 40-60%. [10] also developed an agent-based model for Boston and showed that manual contact tracing with 60% coverage could bring the effective reproduction number below 1. More recently, [11] derived generic formulae for the effective reproduction number under contact tracing and proved that tracing delay and coverage are the dominant factors controlling spread. [12] developed an SVEIHR model to evaluate how vaccination, hospitalization, and contact tracing interact in the transmission dynamics of COVID-19 in Nigeria.

The pandemic spread of COVID-19 in selected countries after the outbreak of SARS-CoV-2 in Wuhan City, China was considered by [14]. The authors estimated the infection rate and the initial number of infected individuals by using officially reported data from the early stages of the epidemic, employing the SIQRK model — Susceptible (S), Infectible (I), Quarantine (Q), and officially confirmed Recovered (RK) populations. The disease was expected to subside about six months after the first patients were found. [15] examined the emergence of COVID-19 in Wuhan, Hubei Province, China, in December 2019, and its spread to the rest of China and overseas. The authors employed population flow data between Wuhan and other cities/regions by 23 January 2020, and used maximum log likelihood estimation and the Akaike Information Criterion (AIC) to estimate a total of 8,679 (95% CI: 7,701–9,732) COVID-19 cases in Wuhan by that date. The study concluded that a comprehensive analysis based on different data sources is necessary to overcome biases inherent in any single source.

A new COVID-19 model with optimal control analysis incorporating a vaccination class was presented by [16]. The authors formulated a vaccine-free model, established stability results for $R_0 < 1$, and extended the model to include vaccination. Four control strategies were considered: prevention, vaccination control, rapid screening of exposed individuals, and identification of infected individuals without prior screening. Using the forward-backward Runge-Kutta method, the results demonstrated that optimal control strategies can significantly minimize infected individuals and improve population health outcomes. [17] carried out a mathematical analysis of COVID-19 via a new mathematical model that included a resistive class together with a quarantine class. The developed model consisted of four compartments: the susceptible class, the healthy (resistive) class, the infected class, and the quarantine class. The authors derived basic properties including boundedness and positivity within a biologically feasible region, and computed the threshold quantity to discuss local equilibria. Numerical simulations were

compared with available real data for Wuhan city, where the infection was initially originated.

A landmark study on quantifying SARS-CoV-2 transmission and the role of digital contact tracing was conducted by [13]. Published in Science, the study used mathematical modeling to determine whether contact tracing could control the COVID-19 epidemic. The authors found that a large proportion of COVID-19 transmission occurred before symptom onset (pre-symptomatic transmission), making traditional manual contact tracing insufficient on its own. They concluded that instantaneous digital contact tracing via smartphone applications would be essential to achieve epidemic control, particularly when combined with case isolation and quarantine. The study estimated that the basic reproduction number (R_0) ranged between 2 and 3, and that contact tracing would need to be near-instantaneous to be effective given the short serial interval of the virus. [18] examined the effectiveness of isolation and contact tracing in managing COVID-19 outbreaks in the United Kingdom. Using a stochastic individual-based transmission model, the authors evaluated the probability of outbreak control under different levels of contact tracing and self-isolation. Their results showed that combined strategies including contact tracing, self-isolation, and physical distancing could substantially reduce the reproduction number. However, the study found that contact tracing alone was insufficient unless more than 80% of contacts were identified and isolated promptly. The findings strongly supported the use of complementary non-pharmaceutical interventions (NPIs) alongside contact tracing programs. [3] investigated the feasibility of controlling COVID-19 outbreaks through isolation and contact tracing. Using a stochastic branching process model, the authors simulated COVID-19 outbreak scenarios under varying assumptions of transmissibility, tracing efficiency, and proportion of cases traced. The study found that the probability of outbreak containment was strongly dependent on the speed of contact tracing and the fraction of contacts successfully traced. For high-transmission scenarios ($R_0 > 2.5$), the model estimated that contact tracing alone had limited success even when 80% of contacts were traced. The study emphasized the need for rapid identification and a robust public health infrastructure to make contact tracing effective.

[8] analyzed the impact of delays on the effectiveness of contact tracing in COVID-19. Using a mathematical model, the authors demonstrated that testing and tracing delays of even a few days could substantially reduce the effectiveness of contact tracing programs. Their study showed that same-day test-trace strategies could reduce the effective reproduction number from values above 1 to values below the epidemic threshold, while delays of more than three days severely diminished the impact. These findings highlighted the importance of rapid and efficient testing frameworks as a prerequisite for successful contact tracing. [10] modeled the impact of testing, contact tracing, and household bubbles on the deconfinement period of the COVID-19 epidemic in Catalonia, Spain. Using a detailed age-structured compartmental model combined with mobility data, the authors found that maintaining an effective reproduction number below 1 after lockdown required a combination of moderate contact reductions, extensive testing, and contact tracing. The study underscored the role of household and workplace transmission dynamics in shaping the overall epidemic and found that targeting high-risk settings through contact tracing could yield disproportionate epidemiological benefits.

While quarantine and contact tracing have been incorporated in several COVID-19 transmission models, the limitation of many previous studies is that they either focused primarily on exposed-class frameworks (e.g., SEIR-type models), emphasized optimal control strategies, vaccination effects, or evaluated tracing effectiveness through stochastic and agent-based approaches without explicitly distinguishing the different epidemiological consequences of tracing susceptible individuals versus tracing infected individuals within a treatment framework.

The present study addresses this gap by formulating an SQIT (Susceptible–Quarantine–Infected–Treated) deterministic model in which contact tracing acts through two distinct pathways: susceptible individuals identified through tracing are transferred into quarantine, while infected individuals identified through tracing are moved directly into treatment. This modelling structure allows the simultaneous investigation of household quarantine, treatment, and differential contact tracing mechanisms within a unified framework. From a mathematical perspective, the study establishes the positivity and boundedness of solutions, derives the disease-free and endemic equilibrium states, computes the basic reproduction number, establishes local stability of the disease-free equilibrium using the Jacobian and Routh–Hurwitz criteria and quantifies parameter influence through normalized sensitivity analysis.

The epidemiological insight generated by this framework is that the effectiveness of contact tracing depends not only on its coverage but also on the pathway through which traced individuals are managed. The results demonstrate that tracing susceptible individuals increases quarantine uptake and interrupts transmission chains, whereas tracing infected individuals accelerates treatment initiation and shortens the infectious period. Sensitivity analysis further identifies the transmission rate, recruitment rate, and contact tracing parameters as dominant determinants of disease spread. Consequently, the findings suggest that combining rapid contact tracing with prompt quarantine and treatment is more effective in reducing COVID-19 transmission than interventions targeted solely at infectious individuals. Unlike previous studies that primarily investigated quarantine and contact tracing independently or within SEIR-based frameworks, this study distinguishes the effects of tracing susceptible and infected individuals in an SQIT structure and examines how these pathways jointly influence transmission dynamics, treatment uptake, and disease control. The aim of this study is to assess the impact of contact tracing and household quarantine on COVID-19 transmission using an SQIT compartmental model.

This study is organized as follows: Section 2 gives the model description, including the model's positivity and boundedness. Section 3 explains an in-depth mathematical analysis of the model, such as the basic reproduction number, local stability analysis of the disease-free equilibrium and also sensitivity analysis of the basic reproduction number. Section 4 gives the numerical simulation of the model, including results and discussions while Section 5 gives the concluding remarks of the findings.

2. Mathematical Methodology

2.1. Model Description and formulation

In this research, an (S, Q, I, T) epidemic model which focuses only on the dynamics of covid-19 outbreak. The model was formulated by splitting the total human population at time t , into four mutually exclusive compartment which are the susceptible $S(t)$, Quarantine $Q(t)$, Infected $I(t)$, and Treated population, with sizes denoted by $S(t)$, $Q(t)$, $I(t)$, and $A(t)$, respectively such that $N = S + Q + I + A$.

At susceptible class, Λ is the total population of a particular country and ε , are the influx of immigrants that enter into the neighborhood. A fraction of immigrants is susceptible at rate (ρ) Newly infected individuals infect susceptible at a rate (β) . During the COVID-19 lock down, those that refuses to adhere to rule of lock down by the government by staying indoor are traced via contact trace at rate (c_1) while some died naturally. At quarantine class, fraction of immigrant is quarantine at rate $(1-\rho)$. Those from susceptible class that refuses to obey lock down rule are traced and put under quarantine for isolation. Those that do not develop symptoms (Asymptomatic) returns back to susceptible at rate (ϕ) . Some also died naturally. Those that are symptomatic via contact tracing progressed to infected class immediately at rate (c_2) . At treated class, quarantined that develops symptoms are adequately undergoing treatment at rate (π) . The infected population also progressed immediately to treated class at rate (ω) . While at infected and treated classes, Some died naturally (μ) and some died due to disease death rate at (δ) . Some among the treated class became recovered and are allowed to be integrated back and join the susceptible due to vaccines wane-off at the rate of ϕ . Thus, considering all the above assumptions made, the model is presented by the following deterministic system of non-linear differential equations which capture the complex interactions between the compartments.

$$\left. \begin{aligned} \frac{dS}{dt} &= \Lambda + \varepsilon\rho - \beta SI - \mu S - c_1 S + \sigma Q + \phi T \\ \frac{dQ}{dt} &= \varepsilon(1 - \rho) + c_1 S - \mu Q - \sigma Q - \pi Q \\ \frac{dI}{dt} &= \beta SI - \mu I - \delta I - c_2 I - \omega I \\ \frac{dT}{dt} &= \pi Q + c_2 I + \omega I - \mu T - \phi T \end{aligned} \right\} \quad (1)$$

$$S(0) = S_0 > 0, I(0) = Q_0 > 0, I(0) = I_0 > 0, T(0) = T_0 > 0 \quad (2)$$

Table 1: Definition of Parameters/Variables

Variables/Parameters	Description
Λ	Total population in a country
ε	Immigrant that comes into a new neighborhood
βSI	Contact rate of infected and susceptible
μ	Natural death rate
$c_1 S$	Contact trace of susceptible that do not stay at home during lockdown
σQ	Quarantine that did not develop symptoms
πQ	Quarantine that develop symptoms
δ	Disease death rate of infected people
c_2	Contact trace of infected people that moved to treated class
ϕ	Recovered class that finally become susceptible
ω	The progressive rate of moving infected to treated
$\varepsilon(1 - \rho)$	Fraction of the immigrant that are quarantined
ρ	Fraction of the immigrant that are susceptible

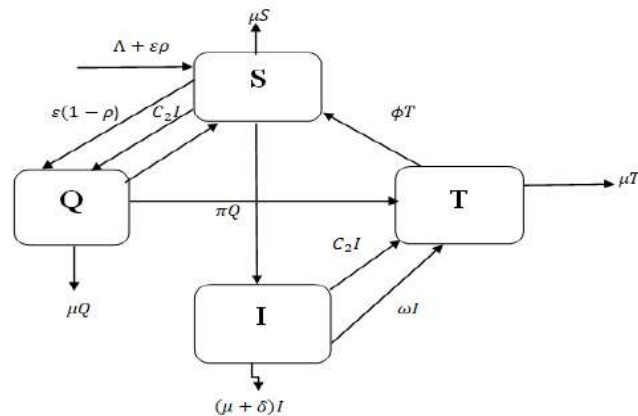


Figure 1: Schematic Diagram of the SQIT COVID-19 epidemic Model

2. 2 Positivity and boundedness of the model

In this section we shall show from model (1) that the state variables are non-negative and the solutions remain positive for all $t \geq 0$. Hence, the parameters in the model are assumed to be positive.

Theorem 1: Let the initial conditions or values of the state variables be such that $\{(S(0) \geq 0, Q(0) \geq 0, I(0) \geq 0, T(0) \geq 0), \in \Omega\}$, then the set $(S(t), I(t), A(t), T(t))$ is non-negative in Ω for all $t \geq 0$.

Considering the first compartment of (1)

$$\frac{dS}{dt} \geq -(\beta I + \mu + c_1)S$$

Using separable variable

$$\frac{dS}{S} \geq -(\beta I + \mu + c_1)dt$$

Integrate both sides

$$\int \frac{dS}{S} \geq - \int (\beta I + \mu + c_1)dt$$

$$\ln S(t) \geq -(\beta I + \mu + c_1)t + c$$

$$S(t) \geq e^{-(\beta I + \mu + c_1)t} \cdot e^c \quad \text{where } e^c = S_0$$

$$S(t) \geq S_0 e^{-(\beta I + \mu + c_1)t}$$

$$S(t) = S_0 e^{-(\beta I + \mu + c_1)t}.$$

$$S(0) = S_0$$

Since $S(t) \geq 0$, for all $t > 0$ provided that $S_0 \geq 0$.

Hence, $S(t) \geq 0$

Considering also the second compartment

$$\frac{dQ}{dt} = \varepsilon(1 - \rho) + c_1S - (\mu + \sigma + \pi)Q$$

$$\frac{dQ}{dt} \geq -(\mu + \sigma + \pi)Q$$

Using separable variable

$$\frac{dQ}{Q} \geq -(\mu + \sigma + \pi)dt$$

Integrate both sides

$$\int \frac{dQ}{Q} \geq - \int (\mu + \sigma + \pi) dt$$

$$\ln Q \geq -(\mu + \sigma + \pi)t + c$$

$$Q(t) \geq e^{-(\mu + \sigma + \pi)t} \cdot e^c \quad \text{where } e^c = Q_0$$

$$Q(t) \geq Q_0 e^{-(\mu + \sigma + \pi)t}$$

$$Q(t) = Q_0 e^{-(\mu + \sigma + \pi)t}$$

$$Q(0) = Q_0$$

Since $Q(t) \geq 0$, for all $t > 0$ provided that $Q_0 \geq 0$.

Hence, $Q(t) \geq 0$

It is possible to show using the same procedure for every other state variable,

$$S(t) \geq S_0 e^{-(\beta I + \mu + c_1)t}, Q(t) \geq Q_0 e^{-(\mu + \sigma + \pi)t} \geq 0, I(t) \geq T(0) e^{-(\mu + \delta + c_2 + \omega)t} \geq 0, T(t) \geq T(0) e^{-(\mu + \phi)t} \geq 0$$

This shows that all the solutions of equation (1) are positive for all $t \geq 0$. Therefore, the COVID-19 transmission model stated in (1) is both epidemiologically significant and numerically well posed in an attainable given region $\Omega \geq 0$

Theorem 2: Every solution in the region $\Omega = \{S(t), Q(t), I(t), T(t) \in \Omega^4 : N(t) \leq \frac{D}{\mu} \text{ positively invariant with respect to the covid-19 model in the population.}$

The solutions for the system are contained and remain in the region Ω for all time $t \geq 0$.

Proof: Considering the equation of the model and adding up all the derivatives with respect to time t , we obtained. $N(t) = S(t) + Q(t) + I(t) + T(t)$

$$\begin{aligned} \frac{dN(t)}{dt} = & \Lambda + \varepsilon \rho - \beta SI - \mu S - c_1 S + \sigma Q + \phi T + \varepsilon - \varepsilon \rho + c_1 S - \mu Q - \sigma Q - \pi Q + \beta SI \\ & - \mu I - \delta I - c_2 I - \omega I + \pi Q + c_2 I + \omega I - \mu T - \phi T \end{aligned}$$

$$\frac{dN(t)}{dt} = \Lambda + \varepsilon - (S + Q + I + T)\mu - \delta I$$

$$\text{Let } \Lambda + \varepsilon = D \text{ and } (S + Q + I + T) = N$$

$$\frac{dN(t)}{dt} = D - \mu N$$

$$I.F = e^{\int \mu dt} = e^{\mu t}$$

$$(e^{\mu t}) \frac{dN}{dt} + e^{\mu t} (\mu N) = D e^{\mu t}$$

$$\frac{d}{dt} (\mu N e^{\mu t}) = D e^{\mu t}$$

Integrating both sides

$$\int \frac{d}{dt} (\mu N e^{\mu t}) = \int D e^{\mu t}$$

$$N e^{\mu t} = \frac{D}{\mu} e^{\mu t}$$

$$N = \frac{D}{\mu} + k e^{-\mu t}$$

$$N(0) = N_0$$

$$N_0 = \frac{D}{\mu} + k$$

$$k = N_0 - \frac{D}{\mu}$$

$$N = \frac{D}{\mu} + \left[N_0 - \frac{D}{\mu} \right] e^{-\mu t}$$

$$N = \frac{D}{\mu} \text{ as } t \rightarrow \infty$$

As $t \rightarrow \infty$ the carrying capacity also reach a constant value

Where N_0 is the initial size of the population.

Therefore,

$$\lim_{t \rightarrow \infty} N(t) \leq \frac{\pi}{\mu} \quad (2)$$

This result implies that the COVID-19 model (1) has non-zero negative and bounded solution in the region Ω and all the solutions starting in Ω approach, enter or stay in Ω . Hence, it is sufficient to conclude that the model is epidemiologically well posed.

3. Mathematical Analysis of the Model

3.1.0 Disease free equilibrium point (DFE) represents a state where the disease is eliminated from the population, and the system reaches a steady state with zero infected individuals.

$$\frac{dS}{dt} = \Lambda + \varepsilon\rho - \beta SI - \mu S - c_1 S + \sigma Q + \phi T$$

$$\frac{dQ}{dt} = \varepsilon(1 - \rho) + c_1 S - \mu Q - \sigma Q - \pi T$$

$$\frac{dI}{dt} = \beta SI - \mu I - \delta I - c_2 S - \omega I$$

$$\frac{dT}{dt} = \pi Q + c_2 I + \omega I - \mu T - \phi T$$

At equilibrium point,

$$\frac{dS}{dt} = \frac{dQ}{dt} = \frac{dI}{dt} = \frac{dT}{dt} = 0 \quad (3)$$

$$\Lambda + \varepsilon\rho - \beta SI - \mu S - c_1 S + \sigma Q + \phi T = 0$$

$$\varepsilon(1 - \rho) + c_1 S - \mu Q - \sigma Q - \pi T = 0$$

$$\beta SI - \mu I - \delta I - c_2 S - \omega I = 0$$

$$\pi Q + c_2 I + \omega I - \mu T - \phi T = 0$$

At DFE, we set $S \neq 0, Q = 0, I = 0, T = 0$ and solving (1), we obtained the DFE as

$$E^o = (S^o, Q^o, I^o, T^o) = \left(\frac{\Lambda + \varepsilon\rho}{\mu + c_1}, 0, 0, 0 \right) \quad (4)$$

3.1.1 Endemic equilibrium points represents a steady state where the disease persists in the population, with a constant number of infected individuals. At endemic equilibrium, we set $S \neq 0, Q \neq 0, I \neq 0, T \neq 0$, thus

$$\Lambda + \varepsilon\rho - \lambda^* S^* - k_1 S^* + \sigma Q^* + \phi T^* = 0 \quad (5)$$

$$\varepsilon(1 - \rho) + c_1 S^* - k_2 Q^* = 0 \quad (6)$$

$$\lambda^* - k_3 I^* = 0 \quad (7)$$

$$\pi Q^* + k_4 I^* - k_5 T^* = 0 \quad (8)$$

Where,

$$\lambda^* = \beta I^*$$

$$k_1 = (\mu + C)$$

$$k_2 = (\mu + \sigma + \pi)$$

$$k_3 = (\mu + \delta + c_2 + \omega)$$

$$k_4 = (c_2 + \omega)$$

$$k_5 = (\mu + \phi)$$

Solving equations 15-18 simultaneously, we obtained the endemic equilibrium points of the covid-19 model represented by $x^* = (S^*, Q^*, I^*, T^*)$ and obtained as:

$$S^* = \frac{(k_2 k_3 k_5) \Lambda + (k_2 k_3 k_5) \varepsilon \rho}{(k_2 k_3 k_5) (\lambda^* + k_1) - \sigma (\varepsilon (1 - \rho) + c_1 S^*) k_3 k_5 + \phi S^* (\varepsilon (1 - \rho) + c_1 (k_3 \pi) + k_2 k_4 \lambda^*)} \quad (9)$$

$$Q^* = \frac{\varepsilon (1 - \rho) + c_1}{k_2} \left(\frac{(k_2 k_3 k_5) \Lambda + (k_2 k_3 k_5) \varepsilon \rho}{(k_2 k_3 k_5) (\lambda^* + k_1) - \sigma (\varepsilon (1 - \rho) + c_1 S^*) k_3 k_5 + \phi S^* (\varepsilon (1 - \rho) + c_1 (k_3 \pi) + k_2 k_4 \lambda^*)} \right) \quad (10)$$

$$I^* = \frac{\lambda^*}{k_3} \left(\frac{(k_2 k_3 k_5) \Lambda + (k_2 k_3 k_5) \varepsilon \rho}{(k_2 k_3 k_5) (\lambda^* + k_1) - \sigma (\varepsilon (1 - \rho) + c_1 (k_3 k_5) + \phi (\varepsilon (1 - \rho) + c_1 (k_3 \pi) + k_2 k_4 \lambda^*))} \right) \quad (11)$$

$$T^* = \frac{\pi}{k_5} \left(\frac{(k_2 k_3 k_5) \Lambda + (k_2 k_3 k_5) \varepsilon \rho}{(k_2 k_3 k_5) (\lambda^* + k_1) - \sigma (\varepsilon (1 - \rho) + c_1 S^*) k_3 k_5 + \phi (\varepsilon (1 - \rho) + c_1 (k_3 \pi) + k_2 k_4 \lambda^*)} \right) \quad (12)$$

3.2 Basic Reproduction Number (R_0)

The computation of basic reproduction number is essential and this is defined as defined as the effective number of secondary infections caused by infected individuals in his entire period of infectiousness. To determine the next generation matrix for the model, the number of ways that new infectious can be transformed between compartment are put into considerations. F_1 and V_1 are computed as follows, using the approach of [23], where

$$f_i = \begin{pmatrix} 0 \\ \beta SI \end{pmatrix} \quad (13)$$

$$v_i = \begin{pmatrix} k_2 Q \\ K_3 I \end{pmatrix} \quad (14)$$

Where,

$$k_2 = (\mu + \sigma + \pi),$$

$$k_3 = (\mu + \delta + c_2 + \omega)$$

Let k be the next generation matrix, comparing of two parts F and v^{-1} Where,

$$F = \left[\frac{\partial f_1(x_0)}{\partial x_j} \right], \quad V = \left[\frac{\partial V}{\partial x_j} \right] \quad (15)$$

The term f_i and v_i are computed as

$$F = \begin{bmatrix} \frac{\partial f_1}{\partial Q} & \frac{\partial f_1}{\partial I} \\ \frac{\partial f_2}{\partial Q} & \frac{\partial f_2}{\partial I} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \beta S \end{bmatrix} \quad (16)$$

$$V = \begin{bmatrix} \frac{\partial v_1}{\partial Q} & \frac{\partial v_1}{\partial I} \\ \frac{\partial v_2}{\partial Q} & \frac{\partial v_2}{\partial I} \end{bmatrix} = \begin{bmatrix} k_2 & 0 \\ 0 & k_3 \end{bmatrix} \quad (17)$$

$$V^{-1} = \begin{bmatrix} \frac{1}{k_2} & 0 \\ 0 & \frac{1}{k_3} \end{bmatrix} \quad (18)$$

$$FV^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \beta S \end{bmatrix} \begin{bmatrix} \frac{1}{k_2} & 0 \\ 0 & \frac{1}{k_3} \end{bmatrix} \quad (19)$$

$$FV^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\beta S}{k_3} \end{bmatrix} \quad (20)$$

The basic R_0 which is the dominate eigenvalues of the product FV^{-1} is therefore obtain as $R_0 = \frac{\beta S}{k_3}$ but at dfe, $x_0 = \left(\frac{\Lambda + \varepsilon \rho}{\mu + c_1}, 0, 0 \right)$.

Therefore,

$$R_0 = \frac{\beta(\Lambda + \varepsilon \rho)}{k_3(\mu + c_1)} \quad (21)$$

Where $k_3 = (\mu + \delta + c_2 + \omega)$

$$R_0 = \frac{\beta\Lambda + \beta\varepsilon\rho}{\mu^2 + \mu\delta + \mu c_2 + \mu\omega + \mu c_1 + c_1\sigma + c_1c_2 + \omega c_1} \quad (22)$$

3.2. Stability analysis of the Disease- free equilibrium

Theorem 3: The disease-free state is locally asymptotically stable if the basic reproduction number $R_0 < 1$ and unstable if otherwise.

Proof:

We evaluate the Jacobian matrix of the model at the disease- free equilibrium $\left(\frac{\Lambda + \varepsilon \rho}{\mu + c_1}, 0, 0, 0 \right)$

$$J(x_0) = \begin{bmatrix} -G_1 & \sigma & -\beta q & \phi \\ c_1 & -G_2 & 0 & 0 \\ 0 & 0 & \beta q - G_3 & 0 \\ 0 & \pi & G_4 & -G_5 \end{bmatrix} \quad (23)$$

Where, $q = \frac{\Lambda + \varepsilon \rho}{\mu + c_1}$

$$G_1 = (\mu + c_1)$$

$$G_2 = (\mu + \sigma + \pi$$

$$G_3 = (\mu + \delta + c_2 + \omega)$$

$$G_4 = (c_2 + \omega)$$

$$G_5 = (\mu + \phi)$$

The characteristic equation is obtained from

$$\det(\lambda I - J(E_0)) = 0,$$

which can be written in the form

$$\lambda^4 p_0 + \lambda^3 p_1 + \lambda^2 p_2 + \lambda p_3 + p_4 = 0 \quad (24)$$

where

$$p_0 = 1$$

$$p_1 = (-\beta q + G_1 + G_2 + G_3 + G_5)$$

$$p_2 = (-\beta q G_1 - \beta q G_2 - \beta q G_5 - \sigma c_1 + G_1 G_2 + G_1 G_3 + G_1 G_5 + G_2 G_3 + G_2 G_5 + G_3 G_5)$$

$$p_3 = (\beta q \sigma c_1 - \beta q G_1 G_2 - \beta q G_1 G_5 - \beta q G_2 G_5 - \pi \phi c_1 - \sigma G_3 c_1 - \sigma G_5 c_1 + G_1 G_2 G_3 + G_1 G_2 G_5 + G_1 G_3 G_5 + G_2 G_3 G_5)$$

$$p_4 = (\pi \beta \phi q c_1 + \beta q \sigma G_5 c_1 - \beta q G_1 G_2 G_5 - \pi \phi G_3 c_1 - \sigma G_3 G_5 c_1 + G_1 G_2 G_3 G_5)$$

According to the Routh-Hurwitz criteria, all eigenvalues have negative real parts if

$$p_0 > 0, p_1 > 0, p_2 > 0, p_3 > 0, p_4 > 0, p_1 p_2 > p_3, p_1 p_2 p_3 > p_3^2 + p_1^2 p_4 \quad (25)$$

Substituting the expressions for p_0, p_1, p_2, p_3 and p_4 obtained above, we observe that all coefficients are positive whenever $R_0 < 1$.

Furthermore,

$$p_1 p_2 - p_3 > 0, \text{ and}$$

$$p_1 p_2 p_3 - p_3^2 - p_1^2 p_4 > 0.$$

Hence all Routh-Hurwitz conditions are satisfied for $R_0 < 1$.

Therefore, every eigenvalue of $J(E_0)$ has negative real part and the disease-free equilibrium E_0 is locally asymptotically stable. Conversely, when $R_0 > 1$, at least one eigenvalue becomes positive and E_0 loses stability. Thus, E_0 is locally asymptotically stable if and only if $R_0 < 1$.

3.4. Sensitivity Analysis of the basic Reproduction number

This technique assesses how changes in model parameters affect the outcome of a mathematical model. It determines which parameters most influence the model's behavior, enabling researchers to focus on key factors. By evaluating the impact of parameter variations, sensitivity analysis informs decision-making and improves understanding of complex systems. Adapting the methodology introduced by [20], the normalized forward sensitivity index of a variable "K" that depends differentiable on a parameter "e" is defined as:

$$X_e^K := \frac{\partial K}{\partial e} * \frac{e}{K} \quad (29)$$

Following the approach used in [21], the sensitivity indices of the parameters relative to the basic reproduction number (32) is established. Therefore, the normalized forward sensitivity indices of the basic reproduction number (32) associated with parameters, τ , is calculated as

$$X_{\tau}^{R_0} := \frac{\partial R_0}{\partial \tau} * \frac{\tau}{R_0} \quad (30)$$

R_0 is therefore computed with respect to each of the parameters in table 2 and are seen in table 3 below:

Table 2: Parameter values, baseline values and source used in the model.

Parameters	Baseline Values	Sources
β	0.62×10^{-4}	Calculated from demographic data
Λ	0.25	Calculated from demographic data
c_1	0.06	Assumed
c_2	0.07	Assumed
ε	0.5	Flaxman et al 2020
δ	0.006	Verity et al 2020
ρ	0.3	Assumed
ω	0.01	Flaxman et al 2020
σ	0.1	Assumed
π	0.2	Hellewell J. et al 2020
μ	0.00039	World bank
ϕ	0.0135	Okuoghae 2020

Table 3: Sensitivity Indices of R_0 in terms of the Model's parameter baseline values

Parameter	Sensitivity index
Λ	+0.625000000
β	+1.000000000
ω	-0.1157541382
ε	+0.375000000
c_1	-0.9935419771
c_2	-0.8102789675
μ	-0.01097243424
ρ	+0.375000000
δ	-0.06945248293

3.5. Interpretation of Sensitivity Indices

The purpose of sensitivity analysis is to identify the most influential parameters on R_0 , which can inform disease control strategies. Parameters with positive sensitivity index means that an increase in the parameter value leads to an increase in R_0 , making the disease more transmissible. Conversely, parameters with negative sensitivity index imply that an increase in the parameter value leads to a decrease in R_0 , making the disease less transmissible. By identifying the most sensitive parameters, policymakers can target interventions to reduce disease transmission and control outbreaks more effectively.

The sensitivity indices of the basic reproduction number R_0 , relative to its parameters are summarized in table 3. $\beta, \Lambda, \varepsilon, \rho$ are parameters with positive indices

while parameters with negative indices are c_1, c_2, ω , and μ . For instance, $X_{\beta}^{R_0} = +1.0000$ means that increasing or decreasing β by 10% increases or (decreases) R_0 by 10% while $X_{c_2}^{R_0} = -0.8102789675$ means that increasing or (decreasing) R_0 by 8.102789675% as clearly observed from table 3.

4. Numerical Simulations and Discussion

A computational method used to solve complex mathematical models. In this study, numerical simulations are used to visualize and analyze the behavior of the COVID-19 transmission dynamics using Maple 18.0 mathematical software. The simulation demonstrates model equations and reveals the impact of those parameters on the model.

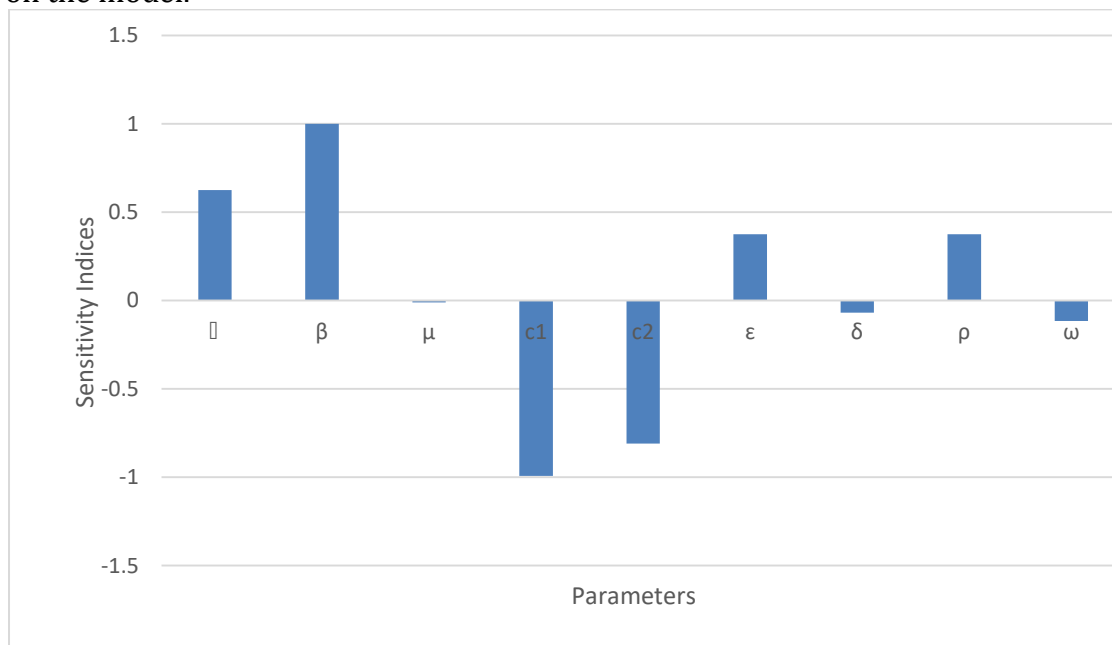


Figure 1: Graphical Representation of the sensitivity indices of R_0

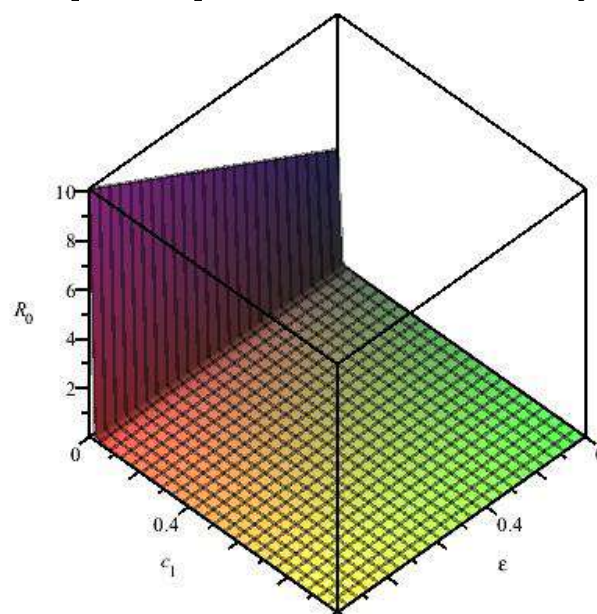


Figure 2: 3-D Plot showing the effect of c_1 and ϵ on the basic reproduction number (R_0)

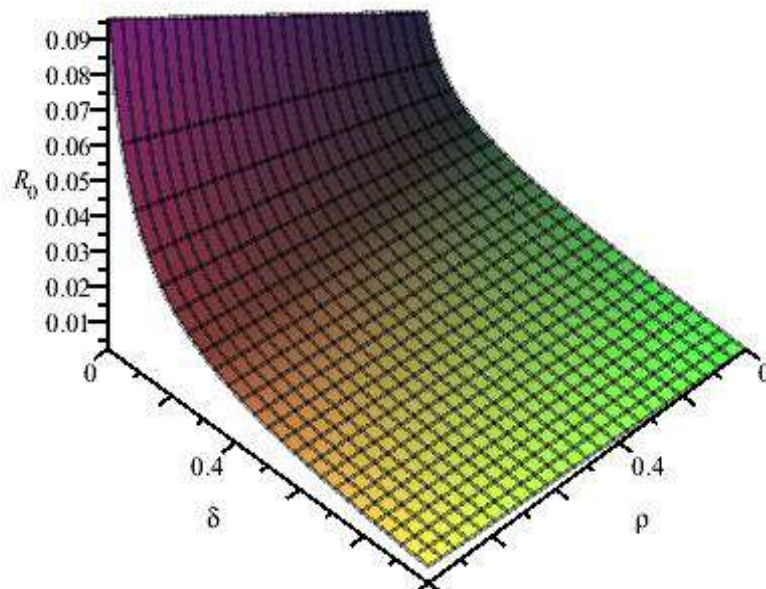


Figure 3: 3-D Plot showing the effect of δ and ρ on the basic reproduction number (R_0)

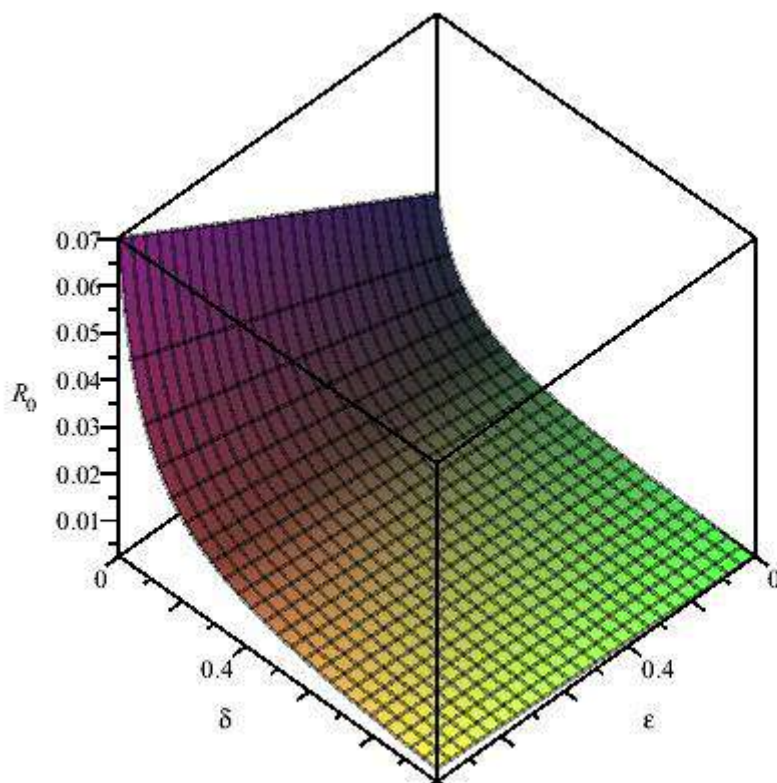


Figure 4: 3-D Plot showing the effect of δ and ϵ on the basic reproduction number (R_0)

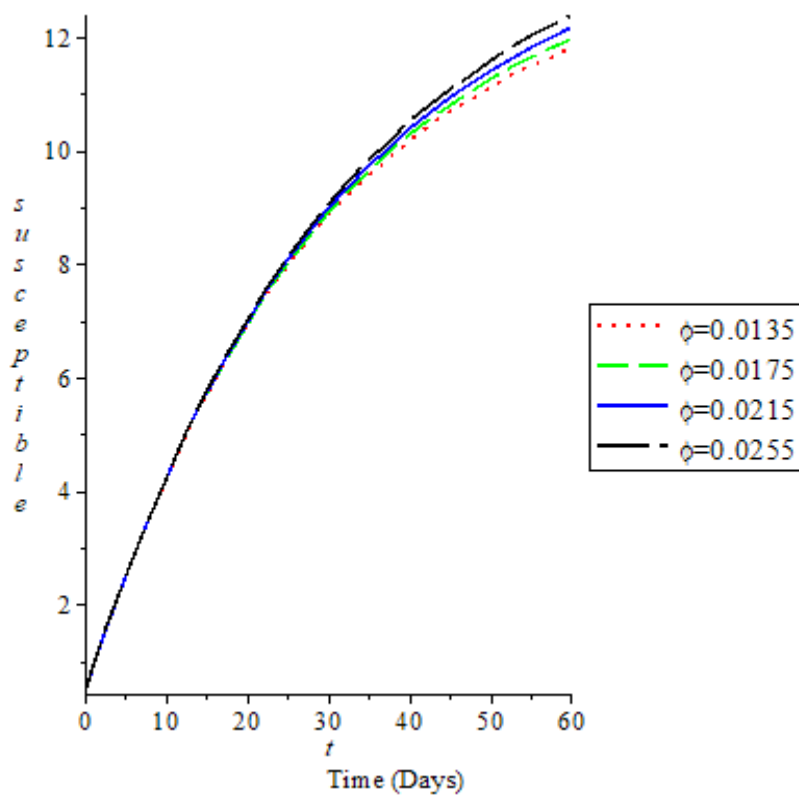


Figure 5: Graph of susceptible population against time for various values of ϕ

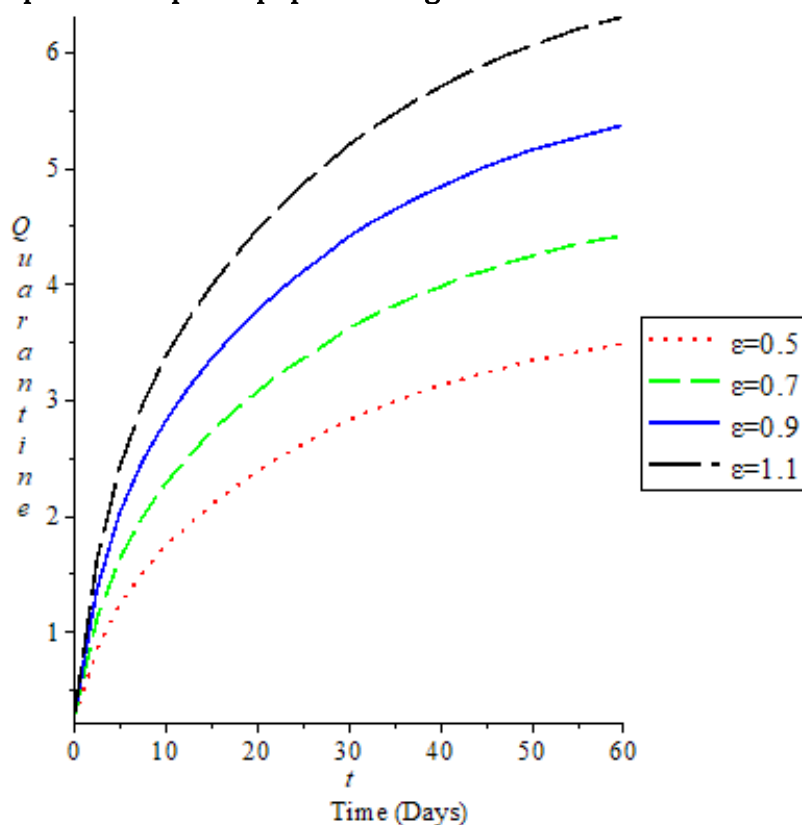


Figure 6: Graph of the Quarantine population against time for various values of ϵ

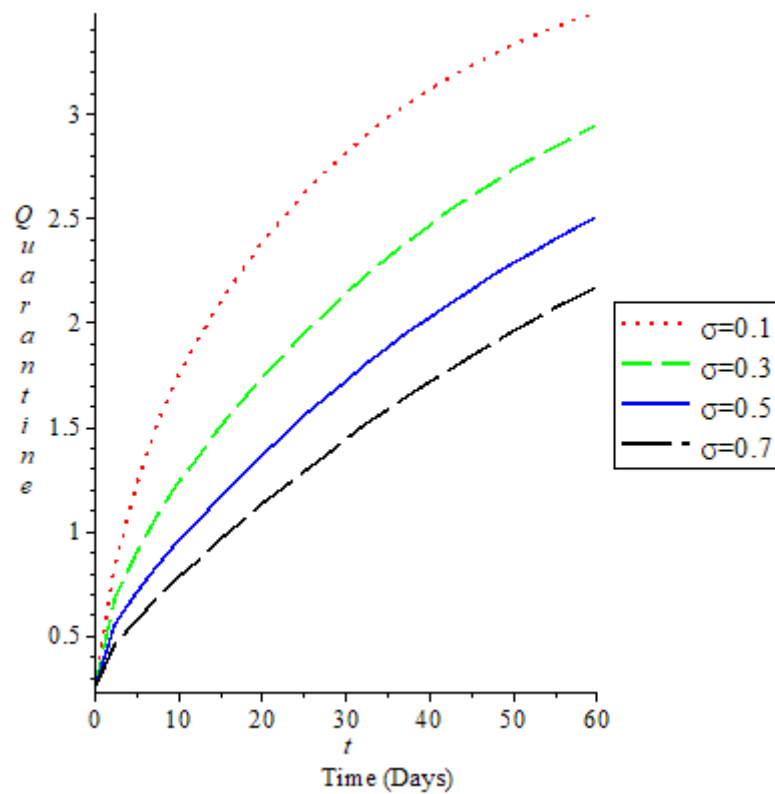


Figure 7: Graph of Quarantine population for various values of sigma (σ)

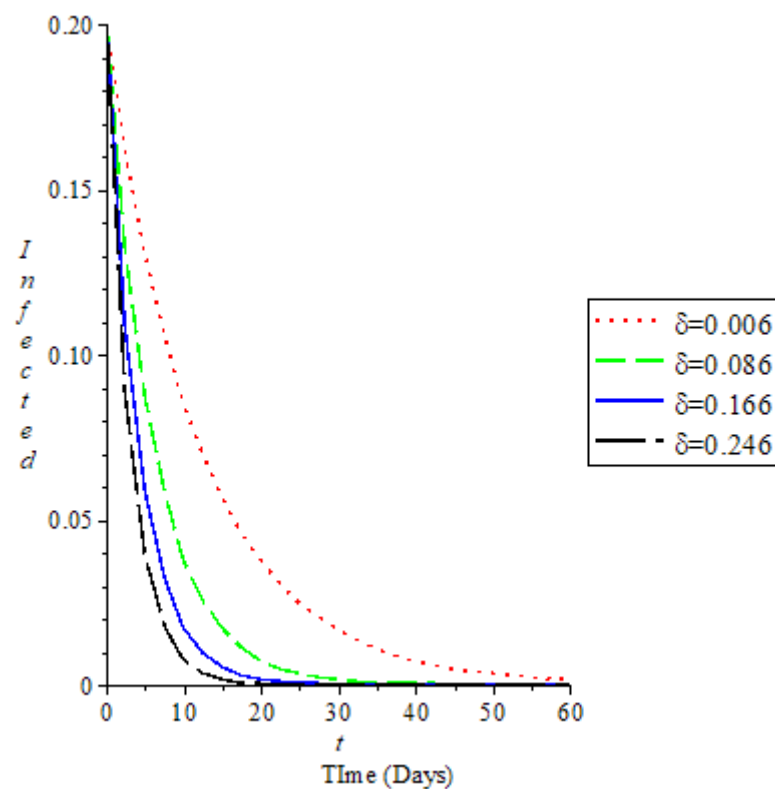


Figure 8: Graph of the Infected against time for various values of delta (δ)

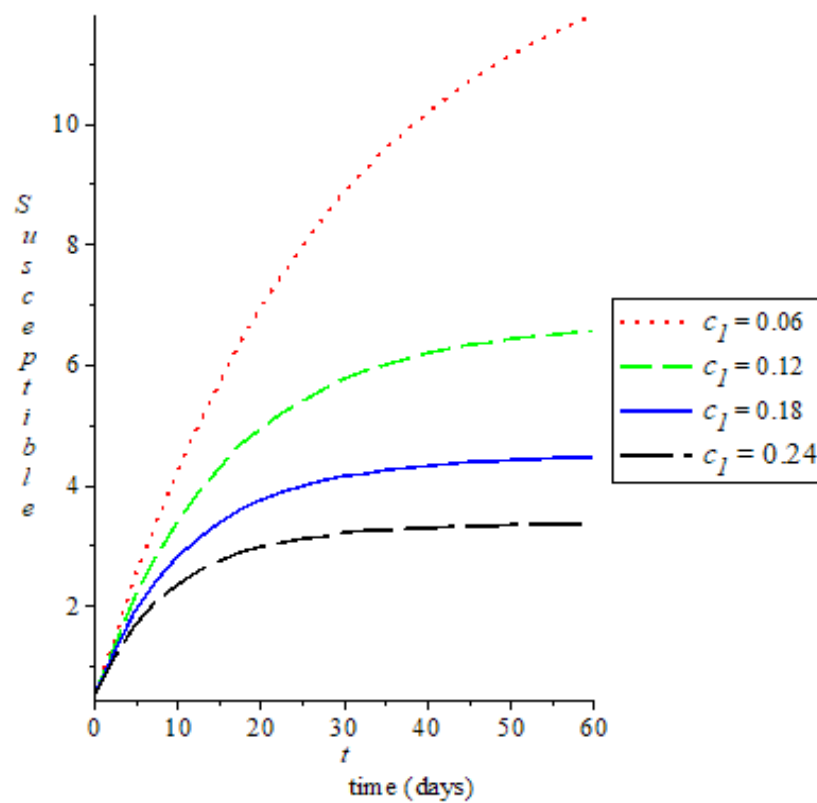


Figure 9: Graph of the Infected against time for various values of c_1 .

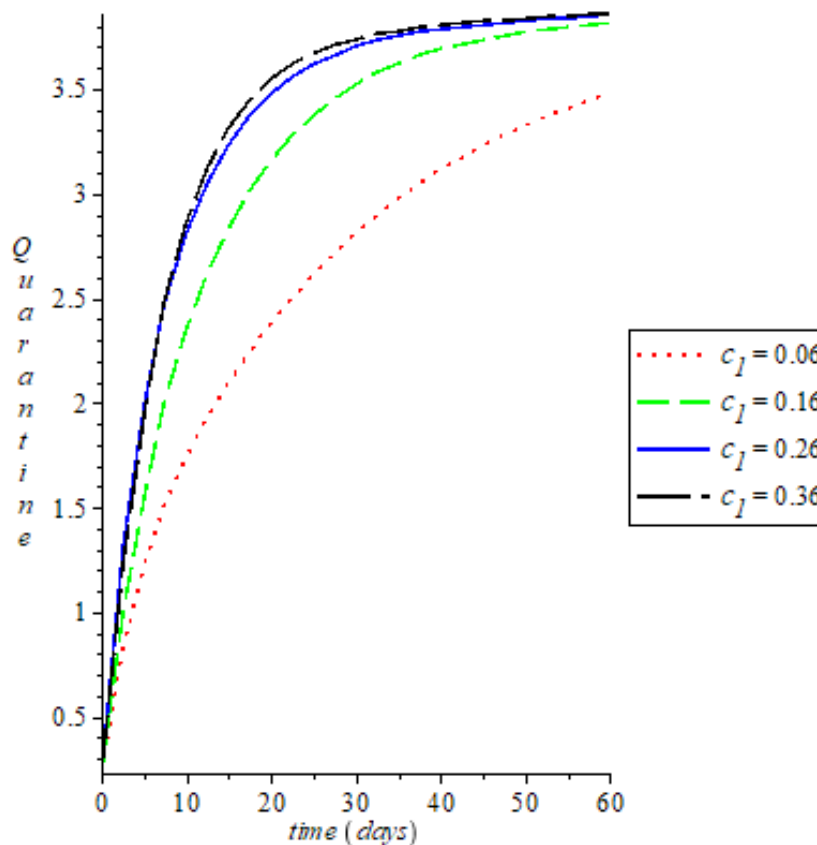


Figure 10: Graph of quarantine population against time for various values of c_1 .

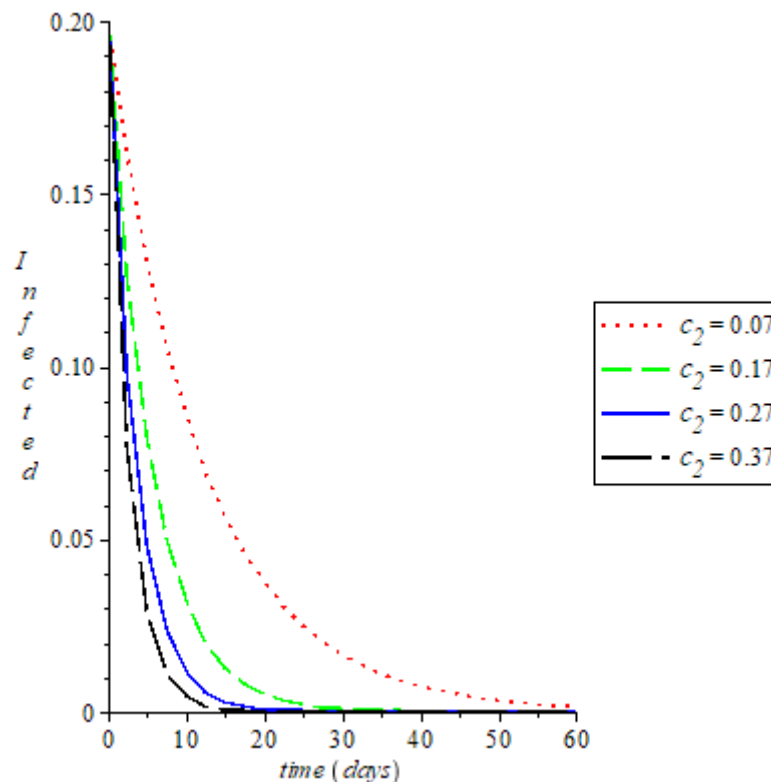


Figure 11: Graph of quarantine population against time for various values of c_2 .

4.2. Discussion

Figure 1: Illustrate the graphical representation of the basic reproduction number in a bar chart. The most sensitive parameters among others are β, c_1, c_2, Λ . It shows the influences of each parameter on the basic reproduction number R_0 . Parameters $\beta, \Lambda, \varepsilon$, and ρ are positively correlated and have a greater impact on R_0 , while parameters ω, c_1, c_2, μ , and δ are negatively correlated with R_0 . R_0 will decrease rapidly as the parameters increase.

Figure 2 reveals the sensitivity of the basic reproduction number to the parameters c_1 and ε . An increase in both c_1 and ε reduces the basic reproduction number. Also, in figure 3, an increase in parameter δ resulted to a decrease in the basic reproduction number while an increase in ρ increases the basic reproduction number.

Figure 4 reveals the sensitivity of the basic reproduction number to the parameters δ and ε . It also shows that an increase in δ decreases R_0 while an increase in ε shows a corresponding increase in parameter ε .

Figure 5 shows that increasing the immunity-loss (or treatment-reversal) parameter ϕ increases the susceptible population over time. Epidemiologically, this indicates that when recovered or treated individuals return more rapidly to the susceptible class, the pool of individuals available for infection is continually replenished. Consequently, disease elimination becomes more difficult because transmission can be sustained even after treatment programs are implemented. This observation is consistent with the findings of Ahmed et al. [17], who reported that maintaining a large susceptible population can prolong disease persistence in compartmental epidemic models.

Figure 6 reveals that increasing (ε) , adds more immigrants directly into quarantine, so the quarantine population grows as ε rises continuously being identified and isolated. Higher (ε) values imply that more people end up in quarantine, showing that ε drives more individuals into quarantine compartment of the model.

Figure 7 shows the graph of quarantine population against time for various values of sigma (σ) . Increase in (σ) , moves asymptomatic quarantined people back to susceptible, thereby decrease quarantine population and adds to susceptible population.

Figure 8: demonstrates that increasing the disease-induced death rate $(\delta\backslash\delta\delta)$ decreases the infected population because infected individuals are removed from the transmission process more rapidly. However, disease-induced mortality is not a desirable control mechanism from a public-health perspective because it reflects severe disease outcomes rather than successful intervention.

In figure 9, An increase in the susceptible contact rate c_1 , leads to a larger quarantine population due to higher transfer from the susceptible compartment while in **figure 10** shows an increase in the infected contact tracing rate c_2 reduces the quarantine population because faster treatment of infected individuals' lower transmission, which in turn reduces the flow of people entering more susceptible are identified and moved into quarantine.

Figure 10 shows that increasing the tracing rate of infected individuals reduces the infected population through earlier treatment initiation. Together, these results indicate that contact tracing acts through two complementary pathways: preventing susceptible individuals from becoming infected and reducing the infectious period of already infected individuals. This finding agrees with the studies of [3], [13], and [8], who demonstrated that rapid and extensive contact tracing is one of the most effective non-pharmaceutical interventions for controlling COVID-19 outbreaks.

5. Conclusion

We analyzed an SQIT compartmental model to assess the role of quarantine, treatment, contact tracing, and disease-induced death in disease transmission. The study established positivity and boundedness of solutions, derived the disease-free and endemic equilibria, computed the basic reproduction number, and examine local stability using the Jacobian matrix and numerical simulations were used to evaluate the influence of model parameters. Sensitivity analysis showed that transmission rate, recruitment rate, and contact tracing rates are the key drivers of the system dynamics. Quarantine and contact tracing reduce disease spread by removing individuals from the transmission chain. Increasing contact tracing for susceptible shifts more individuals into quarantine, while increasing it for infected individuals accelerates movement to treatment and lower the infected population.

Overall, combining contact tracing with quarantine and treatment controls the outbreak more effectively than interventions that target only the infectious period. Overall, the results suggest that combining contact tracing with quarantine and treatment reduces the infected population more than focusing only on treatment during the infectious period.

6. Recommendation

Based on our research work and findings, the followings thrust areas are recommended for future studies:

1. Expand contact tracing to detect cases earlier and reduce transmission

2. Use shorter quarantine periods combined with testing to lower social and economic burden
3. Focus interventions on settings and groups with the highest transmission rates.
4. Improve data collection on contact tracing coverage to strengthen model predictions and policy decisions.

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1. A PROOF OF COLLATZ'S CONJECTURE

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Abstract

The proof of Collatz's Conjecture is divided into three parts. In the first part, it is proved for the set of numbers $\{1, 5, 21, 85, \dots\}$. In the second part, it is proved for odd numbers of the form $(4k+1)$. In the third part, it is shown how numbers of the form $(4k+3)$ iterate into the numbers of the form $(4k+1)$. Thus, the conjecture is proved for all odd 'n'. As even numbers iterate into odd numbers, the conjecture is proved for all positive integers.

Keywords: Proof, Collatz, Conjecture.

Introduction

Consider the following equation for a positive integer 'n':

$$f(n, 1) = \frac{n}{2} \text{ if } n \text{ is even \& } = (3n + 1) \text{ if } n \text{ is odd.}$$

Let, $f(n, r) = f[f(n, r-1), 1]$ for, $r \geq 2$.

Now consider the set,

$$A(n) = \{f(n, 1), \dots, f(n, r), \dots\}$$

Consider an imaginary set.

$$B(n) = \{f(n, 1), \dots, f(n, r), \dots, 1, 4, 2, 1, 4, 2, 1, \dots\}$$

According to Collatz's Conjecture both the sets are identical. That is $A(n) = B(n)$.

Therefore, $A(n) = B(n)$ means Collatz's Conjecture is true for 'n'. It is obvious that, the conjecture has to be proved only for odd values of 'n', because, if 'n' is even, it will iterate into an odd value and ultimately ending in the odd number '1'.

PART 1 : – An analysis of humbers 1, 5, 21, 85, etc

Consider the number, $n = [4^{r-1} + 4^{r-2} + \dots + 1]$

$$f(n, 1) = (4 - 1)[4^{r-1} + \dots + 1] + 1 = 4^r$$

Therefore, $f(n, 2r + 1) = 1$. Hence $A(n) = B(n)$.

Thus, a set $S_1 = \{1, 5, 21, 85, 341, \dots\}$ is generated.

$$S_1 = \{s_r : s_r = \frac{(4^r - 1)}{3}, r \geq 1\}$$

This set is the 'PRIMARY SET', because every other number iterates into a member of this set greater than '1' before iterating into '1'.

PART 2 : – An Analysis Of. $n = (4k + 1)$

As only odd value is to be analysed it can be divided into two parts, that is, $n = (4k + 1)$ & $n = (4k + 3)$. An analysis of, $n = (4k + 3)$, is dealt in Part 3. But Part 2 & Part 3 are interconnected.

Consider $n = (4k + 1)$,

$$f(n, 1) = f[(4k + 1), 1] \\ = 12k + 4$$

$$f(n, 3) = 3k + 1$$

If 'k' is odd, then $f(k, 1) = 3k + 1$.

Or if 'n' is an odd number & $A(n) = B(n)$

Then. $A(4n + 1) = B(4n + 1)$.

Obviously the converse is also true. That is, If 'n' is odd &

$A(4n + 1) = B(4n + 1)$ then, $A(n) = B(n)$.

This is an important result. It also derives the Primary

set, $S_1 = \{1, 1 \times 4 + 1, 5 \times 4 + 1, 21 \times 4 + 1, \dots\}$

$S_1 = \{1, 5, 21, 85, \dots\}$. Similarly

$S_3 = \{3, 13, 53, \dots\}$

$S_7 = \{7, 29, 117, \dots\}$ Or in general,

$S_{2n+1} = \{2n + 1, 8n + 5, 32n + 21, \dots\}$

These sets are important in the understanding of the problem. In these sets, except the first member. all the other members are, obviously, of the $4k + 1$ form.

Consider the following numbers iterations ; -

$3 \rightarrow 5 \rightarrow 1$; $9 \rightarrow 7 \rightarrow 11 \rightarrow 17 \rightarrow 13 \rightarrow 5 \rightarrow 1$.

Now, numbers greater than 9 will be considered.

$f[4k + 1, 1] = 12k + 4$; $f[4k + 1, 3] = 3k + 1$

Let $n = 4k + 1$ & $A(n) = B(n)$; $m \equiv 1 \pmod{3}$ & if,

$m \leq n$, $A(m) = B(m)$. Consider $M = 4[k + r] + 1$,

If $A[3(k + r) + 1] = B[3(k + r) + 1]$, $A(M) = B(M)$

Hence $3(k + r) + 1 \leq 4k + 1$; ie, $3k + 3r \leq 4k$.

Hence, if $3r \leq k$, then $A(M) = B(M)$.

On that basis, The following table has been prepared.

Table: 1

1	2	3	4	5
n	k	m	r	M
13	3	7, 10, 13	1	17 7 (part 3)
17	4	16	1	21
21	5	19	1	25 19 (part 3)
25	6	22, 25	1, 2	29
				33
29	7	28	1, 2	37
33	8	31	1, 2	41 31 (part 3)
37	9	34, 37	1, 2, 3	45
				49
41	10	40	1, 2, 3	53 (13 x 4 + 1)
45	11	43	1, 2, 3	57 43 (part 3)
49	12	46, 49	1, 2, 3, 4	61 23 (part 3)
				65
53	13	52	4	69
57	14	55	4	73 55 (part 3)
61	15	58, 61	4, 5	77

----- 81
65 16 64 5 85

It can be seen from the above table that by PMI,
it is clearly proved that if, $n = 4k + 1$ then,

$$A(n) = B(n).$$

This generates the set $K = \{k: k \in N\}$ & The set

$$C(2^2) = \{n: n = 2^2k + 1 \text{ \& } k \in N\}$$

Though, the set, K itself, PROVES COLLATZ'S

CONJECTURE, the proof is complete only with PART 3.

PART 3 : -An analysis of $n = 4k + 3$

$A(4k + 1) = B(4k + 1)$ has been proved. In this Part 3,

it will be proved, how $n = (4k + 3)$, iterates to $(4k + 1)$.

Let, $n = (4k + 3)$; k , can be odd or even. So,

let, $k = 2k_3$ or $2k_3 + 1$,

Therefor, $n_3 = (8k_3 + 3)$ or $n_{31} = (8k_3 + 7)$.

$$(3 - 3) n_3 = (8k_3 + 3)$$

$$f(n_3, 1) = f(2^3k_3 + 2^2 - 1, 1),$$

$$= 3 \cdot 2^3k_3 + (2^3 + 2^2) - 3 + 1$$

$$f(n_3, 2) = 3 \cdot 2^2k_3 + 2^2 + 2 - 1 = 4(3k_3 + 1) + 1$$

$$\text{Therefor, } A(n_3) = B(n_3).$$

Thus the following set is generatrd.

$$C(2^3) = \{3, 11, 19, \dots\} = \{n_3: n_3 = (2^3k + 3), k \in N\}$$

$n_{31} = 8k_3 + 7$ & k_3 can be odd or even. Therefor let,

$$n_4 = 2^4k_4 + 7, \text{ \& } n_{41} = 2^4k_4 + 15.$$

$$(3 - 4) n_4 = 2^4k_4 + 2^3 - 1$$

$$f(n_4, 1) = 3 \cdot 2^4k_4 + (2^4 + 2^3) - 3 + 1$$

$$f(n_4, 2) = 2^4(3k_4 + 1) + 3. \text{ [} 8k + 3 \text{ form]}$$

$$f(n_4, 4) = 2^2(3^2k_4 + 3 + 1) + 1$$

Therefor, $A(n_4) = B(n_4)$.

Thus the following set is genrrated

$$C(2^4) = \{7, 23, 39, \dots\} = \{n_4: n_4 = (2^4k + 7), k \in N\}$$

In, n_{41}, k_4 can be odd or even. Therefore let,

$$n_5 = 2^5k_5 + 15 \text{ \& } n_{51} = 2^5k_{51} + 31$$

$$(3 - 5) n_5 = 2^5k_5 + 2^4 - 1$$

$$f(n_5, 6) = 4(3^3k_5 + 3^2 + 3 + 1) + 1$$

$$\text{Let, } n_r = 2^r k_r + 2^{r-1} - 1 \text{ \& } n_{r1} = 2^r k_r + 2^r - 1$$

$$(3 - r) n_r = 2^r k_r + 2^{r-1} - 1$$

$$f(n_r, 2r - 4) = 4(3^{r-2}k_r + 3^{r-3} + 3^{r-4} + \dots + 1) + 1$$

Thus, the following set is generated.

$$C(2^r) = \{(2^{r-1} - 1), \dots, [(2^{r-1} - 1) + k2^r], \dots\}$$

$$C(2^r) = \{n_r : n_r = (2^r k + 2^{r-1} - 1), k \in N\}$$

PART 4 : - The Conclusion

All these sets put together constitutes the master
set $C(2)$,

$$C(2) = \{C(2^2)UC(2^3)U \dots UC(2^r) \dots : r \rightarrow \infty\}$$

The set, $C(2) = \{1, 3, 5, 7 \dots\}$

$$C(2) = \{n_2 : n_2 = (2k + 1). k \in N\}$$



And $A(n_2) = B(n_2)$.

The above set represents all odd numbers and as all even numbers iterate to an odd number, for all Positive integers,

$$A(n) = B(n)$$

Thus, COLLATZ'S CONJECTURE , is proved for all positive integers' n' .

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2. A SHORT ESSAY ON OPPERMANN'S CONJECTURE AND ITS IMPACT ON GOLDBACH'S HYPOTHESIS

I am an amateur mathematician. While browsing the Internet, I came across the Conjecture of Oppermann with respect to Prime numbers. As per his conjecture "There is at least one Prime between $[n^2 - n, n^2]$ & There is at least one Prime between $[n^2, n^2 + n]$ "

I looked in to Wikipedia. As per Wikipedia, the Oppermann's Conjecture is more powerful than the Conjectures of Legendre, Brocard and Andrica. As per Legendre's conjecture there is at least one Prime in the interval $[n^2, (n+1)^2]$. As per Oppermann's Conjecture there are at least two Primes. As per Brocard's Conjecture, if p_n is the n th Prime & $p_{(n+1)}$ is the $(n+1)$ th Prime, there are at least four Primes in the interval $[p_n^2, p_{n+1}^2]$. This obviously is true even for twin Primes, if Opperman's Conjecture is true.

As per Andrica's Conjecture, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$.

Now let me restate Oppermann's conjecture in a different way. For every positive integer, $m > 1$, There is at least 1 Prime in the interval $[m^2, m^2 + m]$ And For every positive integer, $m \geq 1$ There is at least 1 Prime in the interval, $[m^2 + m, (m+1)^2]$. Please note $(m+1)^2 - (m+1) = (m^2 + m)$ & and as such both the statements are one and the same.

Let $p_n = m^2 + a$ & $p_{n+1} = (m+1)^2 - b$, and $a = 1$ & $b = 2$, if m is even and $a = 2$ & $b = 1$, if m is odd.

$$\begin{aligned} \sqrt{p_{n+1}} - \sqrt{p_n} &= [(m+1)^2 - b]^{\frac{1}{2}} - [m^2 + a]^{\frac{1}{2}} \\ &< [(m+1)^2]^{\frac{1}{2}} - [m^2]^{\frac{1}{2}} = 1. \end{aligned}$$

Therefore, Andrica's and Opperman's conjectures are one and the same.

I decided to find out the distribution of primes from $m = 1$ onwards, in both the intervals namely, the first interval $[m^2, m^2 + m]$ and the second interval $[m^2 + m, (m+1)^2]$. I went up to 100 and that seem to be more than sufficient to form a conclusion. Giving all the 100 is difficult and so I am shortening as follows.

A1 = (sum of Primes in the first interval from $m-4$ to m)/5

A2 = (sum of primes in the second interval from $m-4$ to m)/5

A = [sum of primes less than $(m+1)^2$]/ $2m$

m	A1	A2	A	$\sqrt{m}$	$l_e m$
5	1.00	1.00	1.00	2.24	1.61
10	2.20	1.60	1.45	3.16	2.30
15	2.60	2.20	1.77	3.87	2.71
20	2.60	3.60	2.10	4.47	3.00
25	3.60	3.80	2.42	5.00	3.22
30	4.00	3.60	2.65	5.48	3.40
35	4.80	4.00	2.90	5.92	3.56
40	5.20	4.80	3.16	6.32	3.69
45	5.20	5.80	3.42	6.71	3.81
50	6.60	5.40	3.68	7.07	3.91

55	6.80	6.40	3.95	7.42	4.00
60	7.20	7.80	4.24	7.75	4.09
65	8.20	6.60	4.48	8.06	4.17
70	9.00	7.20	4.74	8.37	4.25
75	8.60	7.80	4.97	8.66	4.32
80	8.40	9.60	5.23	8.94	4.38
85	9.00	9.40	5.46	9.22	4.44
90	8.60	11.20	5.71	9.49	4.50
95	10.00	10.80	5.95	9.75	4.55
100	11.80	10.20	6.21	10.00	4.61

It is obvious from the above table that as m increases, the number of Primes in each interval not only increases, but it also increases faster than $\sqrt{m}$. The average A also increases faster than $\log m$. I googled for $m = 1000$. Number of Primes less than 1002001 is 78650. Number of Primes in the first Interval is 75 and second interval is 77. Therefore A for $m = 1000$ is, $[78650 \div 2000] = 39.325$, and $\sqrt{m} = 31.62$. Even A is increasing faster than $\sqrt{m}$. Now consider three parabolic equations: (1) $y = x^2$; (2) $y = x^2 + x$; (3) $y = (x + 1)^2$. Let the Straight lines $y = m$, where m is a positive integer Intersect these parabolas at $A(m)$, $B(m)$, & $C(m)$ reply. All the integers lie on these lines. $A(m) = (m, m^2)$; $B(m) = (m, m^2 + m)$; And $C(m) = [m, (m + 1)^2]$. Every Integer lie on the lines AB & BC . Hence all the Primes lie on the lines AB & BC . If, $X(m)$ and $Y(m)$ represent the number of Primes on lines AB & BC , there is nothing surprising that as $m \rightarrow \infty$, $X(m) \rightarrow \infty$ & $Y(m) \rightarrow \infty$. Both of them tend to infinity much faster than $\sqrt{m}$. In addition $\frac{\sum_1^m \{X(m) + Y(m)\}}{2m} \rightarrow \infty$ faster than $\sqrt{m}$, because, the Primes are also infinite. Thus, all these statements can be considered as GREAT UNDERSTATEMENTS.

Goldbach's Hypothesis

As per Goldbach's Hypothesis "every even number Greater than 4, can be expressed as a sum of two odd Primes". The hypothesis can be proved wrong by giving a counter example. It is possible to give a counter example, if and only if a prime, say the n th prime ' p_n ' exists, such that the $(n + 1)$ th prime $p_{(n+1)} > 2p_n$. Let the number of primes between p_n & $\{\sqrt{p_n} + 1\}^2$ be $U(n)$ and number of primes between p_n & $2p_n$ be $V(n)$. $[2p_n - p_n] \div [\{\sqrt{p_n} + 1\}^2 - p_n] = p_n \div [2\sqrt{p_n} + 1] \cong \frac{1}{2} \sqrt{p_n}$. So as $n \rightarrow \infty$, not only $U(n)$ & $V(n) \rightarrow \infty$, but also $\left[\frac{V(n)}{U(n)}\right] \rightarrow \infty$.

Therefore finding a counter example is impossible. I read, a long time ago, somewhere, probably in one, two, three - to infinity of George Gamov, that "it has been proved that all theorems cannot be proved". The Author, further stated in that book that If someone proves that Fermat's Last Theorem was not provable, then that itself constitutes a proof of the theorem because, if it is untrue, then that can be proved by a counter example. Now as it is not possible to disprove Goldbach's Hypothesis, does it constitute a proof of the Hypothesis?

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**3. If $f(a) = 1 - a^p - (1 - a)^p \equiv 0 \pmod{p^3}$, then
 $f(a_r) \equiv 0 \pmod{p^{2r+1}}$ and
 3^{p-1} is not $\equiv 1 \pmod{p^3}$, if $p \equiv 1 \pmod{6}$**

(1) Introduction

Let $\frac{x}{x+y} = a$, & $\frac{y}{x+y} = 1 - a = b$. Then, an analysis of
 $(x + y)^p - x^p - y^p \equiv 0 \pmod{p^3}$ can be analysed by analysing
the equation $1 - a^p - (1 - a)^p \equiv 0 \pmod{p^3}$ --- (1)

That is, $1 - a^p - b^p \equiv 0 \pmod{p^3}$ --- (1)

It has been proved that $2^{p-1} - 1$ is not divisible by p^3
for any prime 'p' Hence '2' is not a solution to eqn (1).

$$f(a) = 1 - a^p - b^p = p \sum P_s \cdot (ab)^{(2s-1)} \cdot (1 - ab)^{(p-6s+3)/2}$$

$$\text{Where } P_s = \frac{2}{p-2s+1} \cdot \frac{p-2s+1}{2} C_{(2s-1)} \text{ --- (2)}$$

(2) If, $(1 - ab) \equiv 0 \pmod{p}$ then $f(a_r) \equiv 0 \pmod{p^{(2r+1)}}$

Let $a \equiv a_1 \pmod{p}$ & $a_1^{p-1} \equiv 1 \pmod{p}$ & not $\equiv 1 \pmod{p^2}$

Then, a_r is defined as, $a_r = (a_{r-1})^p$ --- (3)

Therefore, $a_r^{p-1} \equiv 1 \pmod{p^r}$ & not $\equiv 1 \pmod{p^{r+1}}$

$1 - ab = 1 - a + a^2 = 1 - b + b^2 \equiv 0 \pmod{p}$ & therefore

$$1 - a_r b_r = 1 - a_r + a_r^2 = 1 - b_r + b_r^2 \equiv 0 \pmod{p^r} \text{ --- (4)}$$

From (4), $f(a_r) \equiv 0 \pmod{p^r}$ --- (5)

$1 - a + a^2 \equiv 0 \pmod{p}$ has no solution if $p \equiv 5 \pmod{6}$

So, $p \equiv 1 \pmod{6}$. Hence, In eqn (2) every term on the RHS

are divisible by $p(1 - a_r b_r)^2$ & so $f(a_r) \equiv 0 \pmod{p^{2r+1}}$

It can also be proved in a different way.

From (5), $f(a_{2r+1}) \equiv 0 \pmod{p^{2r+1}}$ & so

$$f(a_{2r+1}) = f(a_r) + [a_{2r+1} - a_r] p [b_r^{p-1} - a_r^{p-1}] + \frac{1}{2!} [a_{2r+1} - a_r]^2 p(p - 1 - b_r p - 2 - a_r p - 2 + \dots) \text{ \& so,}$$

$$\underline{f(a_{2r+1}) \equiv f(a_r) \equiv 0 \pmod{p^{2r+1}}} \text{ --- (6)}$$

(3) $f(a_r) \equiv 0 \pmod{p^{2r+1}}$

$$f(a) = p \sum P_s (ab)^{2s-1} \cdot (1 - ab)^{(p-6s+3)/2} \text{ --- (2)}$$

$$\frac{1}{p} \cdot f(a) = ab (1 - ab)^{(p-3)/2} \cdot \sum P_s \left[\frac{a^2 b^2}{(1 - ab)^3} \right]^{s-1}$$

$$\text{Let } B = ab(1 - ab)^{(p-3)/2}, K = \left[\frac{a^2 b^2}{(1 - ab)^3} \right] \text{ \&}$$

$$\phi(k) = \sum P_s K^{(s-1)}, \text{ Then}$$

$$\frac{1}{p} f(a) = B \phi(K) \text{ --- (7)}$$

In eqn(7), ab is not $\equiv 1 \pmod{p}$, $a \neq 2, -1$ or 2^{-1}

Therefore $B, K, \frac{dB}{da}, \frac{dK}{da}$ are not divisible by "p".

$$\text{Hence } \phi(k_1) \equiv \phi(k_2) \equiv \phi(k_3) \equiv 0 \pmod{p^2} \text{ --- (8)}$$

$$b_1^{p-1} - a_1^{p-1} = \phi(K_1) \frac{dB_1}{da_1} + B_1 \phi'(K_1) \frac{dK_1}{da_1}, \text{ Therefore}$$

$$\phi'(K_1) \equiv 0 \pmod{p} \text{ \& not } \equiv 0 \pmod{p^2} \text{ --- (9)}$$

$$\text{Hence, } [\phi(K_1) \div \phi'(K_1)] \equiv 0 \pmod{p} \text{ --- (10)}$$

$$\begin{aligned}\phi(K_1) &= \frac{1}{p} \frac{f(a_1)}{B_1}, \text{Therefore} \\ \phi'(K_1) \frac{dK_1}{da_1} &= \frac{b_1^{p-1} - a_1^{p-1}}{B_1} - \frac{1}{p} f(a_1) \frac{1}{B_1^2} \frac{dB_1}{da_1}, \text{ \& so,} \\ \phi(K_1) \div \left[\phi'(K_1) \frac{dK_1}{da_1} \right] &= \frac{1}{p} \frac{f(a_1)}{B_1} \div \left\{ \frac{b_1^{p-1} - a_1^{p-1}}{B_1} - \frac{f(a_1)}{p B_1^2} \frac{dB_1}{da_1} \right\} \\ \phi(K_1) \div \left[\phi'(K_1) \frac{dK_1}{da_1} \right] &= f(a_1) \div \left\{ p(b_1^{p-1} - a_1^{p-1}) - \frac{f(a_1)}{B_1} \frac{dB_1}{da_1} \right\} \\ \frac{\phi(K_1)}{\phi'(K_1) \frac{dK_1}{da_1}} \left\{ p(b_1^{p-1} - a_1^{p-1}) - \frac{f(a_1)}{B_1} \frac{dB_1}{da_1} \right\} &= f(a_1) \\ \frac{\phi(K_1)}{\phi'(K_1) \frac{dK_1}{da_1}} p(b_1^{p-1} - a_1^{p-1}) &= f(a_1) + \frac{\phi(K_1)}{\phi'(K_1) \frac{dK_1}{da_1}} \frac{f(a_1)}{B_1} \frac{dB_1}{da_1} \quad \text{--- (12)}\end{aligned}$$

LHS of eqn (12) $\equiv 0 \pmod{p^3}$ and
not $\equiv 0 \pmod{p^4}$. Hence, as $f(a_1) \equiv 0 \pmod{p^3}$
RHS of eqn (12) $\equiv 0 \pmod{p^3}$ & not $\equiv 0 \pmod{p^4}$
The second term of RHS of eqn(12) $\equiv 0 \pmod{p^4}$ --- (13)

$$\frac{\phi(K_2)}{\phi'(K_2) \frac{dK_2}{da_2}} p(b_2^{p-1} - a_2^{p-1}) \equiv f(a_2) + \frac{\phi(K_2)}{\phi'(K_2) \frac{dK_2}{da_2}} \frac{f(a_2)}{B_2} \frac{dB_2}{da_2} \quad \text{--- (14)}$$

LHS of eqn (14) $\equiv 0 \pmod{p^4}$
Therefore, $f(a_2) \equiv 0 \pmod{p^4}$ --- (15)
Therefore, from eqns (12), (13), (14), (15)

$f(a_3) \equiv 0 \pmod{p^5}$ and so
 $f(a_2) \equiv f(a_3) \equiv 0 \pmod{p^5}$ --- (16)
From eqn (16). by PMI, $f(a_r) \equiv 0 \pmod{p^{2r+1}}$ --- (17)

(4) If, $p \equiv 1 \pmod{6}$, 3^{p-1} is not $\equiv 1 \pmod{p^3}$

It is assumed that for a prime, "p"
 $p \equiv 1 \pmod{6}$ & $3^{p-1} \equiv 1 \pmod{p^3}$. --- (18)
If $p \equiv 1 \pmod{6}$, Then solution exists for the equation

$$1 + a + a^2 \equiv 0 \pmod{p}$$

So let, $1 + a_r + a_r^2 \equiv 0 \pmod{p^r}$ --- (19)

$$1 + a_1 + a_1^2 = 1 + (1 - b_1) + (1 - b_1^2) = 3(1 - b_1) + b_1^2$$

$$\text{Hence, } b_1^2 \equiv -3(1 - b_1) \equiv -3a_1 \pmod{p}$$

$$\text{Therefore, } b_1^{2p} \equiv -3^p a_1^p \equiv -3a_2 \equiv b_2^2 \equiv 0 \pmod{p^2}$$

$$\text{Hence, } b_1^p \equiv b_2 \equiv 1 - a_2 \equiv 1 - a_1^p \pmod{p^2}$$

$$\text{So, } (1 - a_1)^p \equiv (1 - a_1^p) \pmod{p^2}$$

$$\text{ie, } f(a_1) = 1 - a_1^p - (1 - a_1)^p \equiv 0 \pmod{p^2} \quad \text{--- (20)}$$

$$\text{Similarly, } f(a_2) \equiv 0 \pmod{p^3} \quad \text{--- (21)}$$

$$\text{So, } f(a) \equiv 0 \pmod{p^3}$$

$$\text{\& hence } f(a_r) \equiv 0 \pmod{p^{2r+1}} \quad \text{--- (22)}$$

$$b_3^2 \equiv -3a_3 \pmod{p^3} \text{ \& So, } b_3^{2p} \equiv -3^p a_3^p \pmod{p^4}$$

$$\text{That is, } b_4^2 \equiv -3^p a_4 \pmod{p^4}$$

$$\text{That is, } -3a_4 \equiv -3^p a_4 \pmod{p^4}$$

$$\text{Therefore, } 3^{p-1} \equiv 1 \pmod{p^4}$$

$$\text{Hence by PMI, } 3^{p-1} \equiv 1 \pmod{p^r} \text{ for all values of } r \quad \text{--- (23)}$$

Eqn (23) is possible if and only if $p = 1$. Therefore

$$\underline{3^{p-1} \text{ is not } \equiv 1 \pmod{p^3} \text{ if } p \equiv 1 \pmod{6}}$$

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RADIAL SCHRÖDINGER EQUATION ON THE EINSTEIN SCHWARZSCHILD MANIFOLD AND FLRW MANIFOLD

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ABSTRACT. In this paper we give different techniques for solving the Schrödinger equation as well as a solution radial developable in Frobenius series on the Einstein Schwarzschild manifold and FLRW manifold.

INTRODUCTION

The Schrodinger equation is the fundamental equation of quantum physics, just as Newton's law is in classical physics. It is used to describe a wide variety of phenomena, whether in quantum optics (laser beam propagation), atomic physics (superconductivity, Bose-Einstein condensation), electronic technology (semiconductors, transistors, memories), plasma physics, astrophysics, electron microscopy, neutronics, chemistry, biology, etc. The equation is written as follows: We denote:

$$\begin{cases} E = h\nu \\ p = \frac{h}{\lambda} \end{cases}$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V\Psi$$

In both expressions above, the letter h denotes Planck's constant and $\hbar = \frac{h}{2\pi}$. We consider the standard Schwarzschild metric of mass M :

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

and we restrict ourselves to the spatial slice $t = \text{const}$, which is a Riemannian manifold (Σ, g) with metric

$$ds_\Sigma^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad r > 2M.$$

We denote

$$f(r) = 1 - \frac{2M}{r}.$$

The stationary Schrödinger equation on (Σ, g) is written:

$$-\frac{\hbar^2}{2m} \Delta_g \Psi + V(r)\Psi = E\Psi.$$

1. RADIAL SCHRÖDINGER EQUATION ON THE EINSTEIN SCHWARZSCHILD MANIFOLD

Separation of Variables. By writing

$$\Psi(r, \theta, \varphi) = R(r)Y_{lm}(\theta, \varphi),$$

we obtain for the radial Laplacian:

$$\Delta_g \Psi = \left[\frac{\sqrt{f}}{r^2} \frac{d}{dr} \left(r^2 \sqrt{f} R'(r) \right) - \frac{l(l+1)}{r^2} R(r) \right] Y_{lm}.$$

We thus obtain the radial equation:

$$(1) \quad -\frac{\hbar^2}{2m} \left[f(r) R''(r) + \left(\frac{2}{r} - \frac{3M}{r^2} \right) R'(r) - \frac{l(l+1)}{r^2} R(r) \right] + V(r) R(r) = E R(r).$$

Equation for the reduced function $u(r) = rR(r)$. We set

$$u(r) = rR(r).$$

Equation (1) then becomes

$$(2) \quad -\frac{\hbar^2}{2m} \left[f(r) u''(r) + \frac{M}{r^2} u'(r) - \frac{M + l(l+1)r}{r^3} u(r) \right] + V(r) u(r) = E u(r).$$

A standard form $u'' + P(r)u' + Q(r)u = 0$ is obtained by isolating u'' :

$$(3) \quad u''(r) + P(r)u'(r) + Q(r)u(r) = 0,$$

with

$$P(r) = \frac{M}{r(r-2M)}, \quad Q(r) = -\frac{M + l(l+1)r}{r^3 f(r)} + \frac{2m}{\hbar^2 f(r)} (V(r) - E).$$

Local analysis near the horizon $r = 2M$. We set

$$x = r - 2M, \quad x \rightarrow 0^+.$$

SO

$$f(r) = \frac{x}{2M} + \mathcal{O}(x^2), \quad P(r) = \frac{1}{2x} - \frac{1}{4M} + \mathcal{O}(x),$$

And

$$Q(r) = \frac{q_{-1}}{x} + q_0 + \mathcal{O}(x),$$

with

$$(4) \quad q_{-1} = -\frac{1 + 2l(l+1)}{4M} + \frac{4mM}{\hbar^2} (V(2M) - E).$$

Frobenius Method. We seek a solution in the form:

$$u(x) = x^s \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0.$$

Substituting this into equation (3), the index equation is:

$$s(s-1) + \frac{s}{2} = 0 \implies s\left(s - \frac{1}{2}\right) = 0.$$

So two exponents:

$$\boxed{s_1 = 0, \quad s_2 = \frac{1}{2}.$$

Series near the horizon.

Regular solution.

$$(5) \quad u(x) = c_0 \left(1 - 2q_{-1}x + \mathcal{O}(x^2) \right), \quad s = 0.$$

Second solution (zeroes at the horizon).

$$(6) \quad u(x) = x^{1/2} \left(c_0 + c_1 x + \mathcal{O}(x^2) \right), \quad s = \frac{1}{2}.$$

Physical Interpretation. Since $R(r) = u(r)/r$:

- $s = 0$: $R(r)$ remains finite and nonzero at the horizon $\Rightarrow$ regular solution.
- $s = 1/2$: $R(r) \sim (r - 2M)^{1/2} \Rightarrow$ the wave function vanishes at the horizon.

The physically relevant solution depends on the imposed boundary conditions (regularity, absorption, radiative conditions in turtle coordinates, etc.).

2. LOCAL SERIES EXPANSION OF THE REGULAR SOLUTION NEAR $r = 2M$

We return to the local variable

$$x := r - 2M, \quad x \rightarrow 0^+.$$

The reduced function $u(r) = rR(r)$ satisfies the ordinary equation

$$u''(r) + P(r)u'(r) + Q(r)u(r) = 0,$$

with

$$P(r) = \frac{M}{r(r - 2M)}, \quad Q(r) = -\frac{M + \ell(\ell + 1)r}{r^3 f(r)} + \frac{2m}{\hbar^2 f(r)}(V(r) - E), \quad f(r) = 1 - \frac{2M}{r}.$$

Coefficient Series. Let us expand P and Q in series in x around $x = 0$:

$$P(r) = \frac{1}{2x} + p_0 + p_1x + p_2x^2 + \mathcal{O}(x^3),$$

$$Q(r) = \frac{q_{-1}}{x} + q_0 + q_1x + q_2x^2 + \mathcal{O}(x^3).$$

The coefficients p_i are directly obtained by expanding P :

$$p_0 = -\frac{1}{4M}, \quad p_1 = \frac{1}{8M^2}, \quad p_2 = -\frac{1}{16M^3}.$$

If we write the expansion of V around $r = 2M$ as

$$V(r) = V_0 + V_1x + \frac{1}{2}V_2x^2 + \frac{1}{6}V_3x^3 + \dots,$$

then we obtain (after expanding Q) the following expressions for the first coefficients q_j :

$$q_{-1} = \frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2}$$

$$q_0 = \frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2}$$

$$q_1 = \frac{32M^3m(MV_2 + V_1) - \hbar^2(2\ell^2 + 2\ell + 3)}{16M^3\hbar^2}$$

$$q_2 = \frac{16M^4m(2MV_3 + 3V_2) + 3\hbar^2(\ell^2 + \ell + 2)}{48M^4\hbar^2}.$$

Frobenius expansion — regular solution ($s = 0$). We seek the regular solution near $x = 0$ in the form

$$u(x) = \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0.$$

By identifying the coefficients of the equation series, we obtain the recurrence relations and, for the first coefficients,

$$(\text{order } x^{-1}): \quad c_0 q_{-1} + \frac{1}{2} c_1 = 0 \implies \boxed{c_1 = -2 q_{-1} c_0},$$

$$(\text{order } x^0): \quad c_0 q_0 + c_1 q_{-1} + 3c_2 - \frac{c_1}{4M} = 0 \implies \boxed{c_2 = \frac{2M(-q_0 + 2q_{-1}^2) - q_{-1}}{6M} c_0},$$

and (continuing the same elimination)

$$\boxed{c_3 = -\frac{1}{90M^2} \left[M^2(24p_1 q_{-1} + 28q_0 q_{-1} - 12q_1 - 8q_{-1}^3) - 2M(q_0 - 3q_{-1}^2) - q_{-1} \right] c_0},$$

and finally (order required for the term in x^4) we obtain the explicit formula ^(*)

$$\boxed{c_4 = -\frac{1}{14} \left(c_0 q_2 + c_1 p_2 + c_1 q_1 + 2c_2 p_1 + c_2 q_0 + c_3 q_{-1} - \frac{3}{4M} c_3 \right)}.$$

The above relations are sufficient to construct the expansion up to order x^4 . Dividing by c_0 defines the normalized coefficients

$$a_n := \frac{c_n}{c_0}, \quad n = 1, 2, 3, 4,$$

which gives:

$$\boxed{a_1 = -2q_{-1}}$$

$$\boxed{a_2 = \frac{2M(-q_0 + 2q_{-1}^2) - q_{-1}}{6M}}$$

$$\boxed{a_3 = -\frac{1}{90M^2} \left[M^2(24p_1 q_{-1} + 28q_0 q_{-1} - 12q_1 - 8q_{-1}^3) - 2M(q_0 - 3q_{-1}^2) - q_{-1} \right]}$$

$$\boxed{a_4 = -\frac{1}{14} \left(q_2 + a_1 p_2 + a_1 q_1 + 2a_2 p_1 + a_2 q_0 + a_3 q_{-1} - \frac{3}{4M} a_3 \right)}.$$

Final Development. Thus, the regular solution around $r = 2M$ is written (for a constant c_0)

$$\boxed{u(x) = c_0 \left[1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \mathcal{O}(x^5) \right]}$$

with a_1, a_2, a_3, a_4 given above and the p_i, q_j specified in terms of $M, \ell, m, \hbar$ and the derivatives of V in $r = 2M$.

Remarks and explicit forms.

- We have already explained above the explicit expressions of q_{-1}, q_0, q_1, q_2 in terms of V_0, V_1, V_2, V_3 . Substituting these quantities into the formulas for a_k gives expressions only in $M, \ell, m, \hbar, E$ and the derivatives $V^{(k)}(2M)$.

- For p_1, p_2 we will use the numerical values:

$$p_1 = \frac{1}{8M^2}, \quad p_2 = -\frac{1}{16M^3}.$$

- The chosen solution corresponds to the exponent $s = 0$ (regular solution, $u(0) \neq 0$, so $R(2M) = u(0)/(2M)$ is finite). The other solution has the exponent $s = \frac{1}{2}$ and includes half-integer powers.

(*) the formula for c_4 is written directly based on the already calculated coefficients c_0, c_1, c_2, c_3 — it can be used immediately to obtain the term in x^4 (replace c_1, c_2, c_3 with the expressions given above).

EXPLICIT EXPRESSIONS OF THE COEFFICIENTS $a_1, \dots, a_4$

We recall the intermediate coefficients (expansions in $x = r - 2M$):

$$\begin{aligned} p_0 &= -\frac{1}{4M}, & p_1 &= \frac{1}{8M^2}, & p_2 &= -\frac{1}{16M^3}, \\ q_{-1} &= \frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2}, \\ q_0 &= \frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2}, \\ q_1 &= \frac{32M^3m(MV_2 + V_1) - \hbar^2(2\ell^2 + 2\ell + 3)}{16M^3\hbar^2}, \\ q_2 &= \frac{16M^4m(2MV_3 + 3V_2) + 3\hbar^2(\ell^2 + \ell + 2)}{48M^4\hbar^2}. \end{aligned}$$

The normalized coefficients $a_n = c_n/c_0$ are then expressed:

$$a_1 = -2q_{-1} = \frac{-16M^2m(V_0 - E) + \hbar^2(2\ell^2 + 2\ell + 1)}{2M\hbar^2}.$$

$$a_2 = \frac{2M(-q_0 + 2q_{-1}^2) - q_{-1}}{6M}.$$

Replacing q_{-1} and q_0 explicitly gives

$$\begin{aligned} a_2 &= \frac{1}{6M} \left\{ 2M \left[-\frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2} \right. \right. \\ &\quad \left. \left. + 2 \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right)^2 \right] \right. \\ &\quad \left. - \frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right\}, \end{aligned}$$

which can be simplified algebraically if desired.

The coefficient a_3 (given previously by a condensed formula) is:

$$a_3 = -\frac{1}{90M^2} \left[M^2(24p_1q_{-1} + 28q_0q_{-1} - 12q_1 - 8q_{-1}^3) - 2M(q_0 - 3q_{-1}^2) - q_{-1} \right].$$

Substituting p_1, q_{-1}, q_0, q_1 we explicitly obtain

$$a_3 = -\frac{1}{90M^2} \left\{ M^2 \left[24 \left(\frac{1}{8M^2} \right) \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right) \right. \right. \\ + 28 \left(\frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2} \right) \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right) \\ - 12 \left(\frac{32M^3m(MV_2 + V_1) - \hbar^2(2\ell^2 + 2\ell + 3)}{16M^3\hbar^2} \right) - 8 \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right)^3 \Big] \\ - 2M \left(\frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2} - 3 \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right)^2 \right) \\ \left. - \frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right\}.$$

Finally, for a_4 we use the compact relation

$$a_4 = -\frac{1}{14} \left(q_2 + a_1p_2 + a_1q_1 + 2a_2p_1 + a_2q_0 + a_3q_{-1} - \frac{3}{4M}a_3 \right).$$

Replacing in this expression $p_1, p_2, a_1, a_2, a_3, q_0, q_1, q_2, q_{-1}$ with the formulas given above, we obtain a_4 entirely as a function of $M, \ell, m, \hbar, E, V_0, V_1, V_2, V_3$. For readability, we can write:

$$a_4 = -\frac{1}{14} \left\{ q_2 + \left(\frac{-16M^2m(V_0 - E) + \hbar^2(2\ell^2 + 2\ell + 1)}{2M\hbar^2} \right) \left(-\frac{1}{16M^3} \right) \right. \\ \left. + a_1q_1 + 2a_2 \left(\frac{1}{8M^2} \right) + a_2q_0 + a_3q_{-1} - \frac{3}{4M}a_3 \right\},$$

with a_1, a_2, a_3 given previously and q_2 equal to

$$q_2 = \frac{16M^4m(2MV_3 + 3V_2) + 3\hbar^2(\ell^2 + \ell + 2)}{48M^4\hbar^2}.$$

3. RADIAL SOLUTION OF THE SCHRÖDINGER EQUATION ON AN FLRW MANIFOLD

Consider the FLRW metric in comoving coordinates $(t, \chi, \theta, \varphi)$:

$$ds^2 = -dt^2 + a(t)^2(d\chi^2 + S_k(\chi)^2d\Omega^2), \quad S_k(\chi) = \begin{cases} \sin \chi, & k = +1, \\ \chi, & k = 0, \\ \sinh \chi, & k = -1. \end{cases}$$

The non-relativistic Schrödinger equation on this manifold is written as

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_g \Psi + V(t, \chi) \Psi,$$

where Δ_g is the Laplace-Beltrami associated with the spatial metric $a(t)^2(d\chi^2 + S_k(\chi)^2d\Omega^2)$. Since $\Delta_g = \frac{1}{a(t)^2} \Delta_h$, we obtain

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m a(t)^2} \Delta_h \Psi + V(t, \chi) \Psi.$$

We seek spherically symmetric solutions $\Psi(t, \chi)$. The radial Laplacian on the spatial section is equal to

$$\Delta_h f(\chi) = \frac{1}{S_k(\chi)^2} \partial_\chi (S_k(\chi)^2 \partial_\chi f(\chi)).$$

Spatial Separation (Radial Mode). We set the spatial eigenproblem

$$(*) \quad \frac{1}{S_k(\chi)^2} \partial_\chi (S_k(\chi)^2 \partial_\chi R(\chi)) + \lambda R(\chi) = 0.$$

Setting $u(\chi) = S_k(\chi)R(\chi)$ we obtain

$$u''(\chi) + \lambda u(\chi) = 0.$$

Therefore

$$u(\chi) = A \cos(\sqrt{\lambda} \chi) + B \sin(\sqrt{\lambda} \chi),$$

and consequently

$$R(\chi) = \frac{1}{S_k(\chi)} \left[A \cos(\sqrt{\lambda} \chi) + B \sin(\sqrt{\lambda} \chi) \right], \quad \lambda \geq 0.$$

Temporal Separation (case $V = 0$). We seek $\Psi(t, \chi) = A(t) R(\chi)$. We obtain

$$i\hbar \frac{A'(t)}{A(t)} = \frac{\hbar^2 \lambda}{2m a(t)^2},$$

SO

$$A(t) = A_0 \exp \left(-i \frac{\hbar \lambda}{2m} \int^t \frac{dt'}{a(t')^2} \right).$$

In order to preserve probabilistic normalization, we choose the factor $a(t)^{-3/2}$:

$$\Psi(t, \chi) = \mathcal{N} a(t)^{-3/2} R(\chi) \exp \left(-i \frac{\hbar \lambda}{2m} \int^t \frac{dt'}{a(t')^2} \right).$$

Final radial solution. We obtain the family of radial solutions ($l = 0$):

$$\Psi_\lambda(t, \chi) = \mathcal{N} a(t)^{-3/2} \frac{A \cos(\sqrt{\lambda} \chi) + B \sin(\sqrt{\lambda} \chi)}{S_k(\chi)} \exp \left(-i \frac{\hbar \lambda}{2m} \int^t \frac{dt'}{a(t')^2} \right).$$

Flat case $k = 0$. Since $S_0(\chi) = \chi$, the regular solution at $\chi = 0$ ($A = 0$) gives:

$$\Psi(t, r) = \mathcal{N} a(t)^{-3/2} \frac{\sin(\kappa r)}{r} \exp \left(-i \frac{\hbar \kappa^2}{2m} \int^t \frac{dt'}{a(t')^2} \right), \quad \kappa = \sqrt{\lambda}.$$

These solutions form a basis for the radial modes (generalized spherical harmonics $l = 0$) on the spatial section of the FLRW metric.

EXPLICIT EXAMPLES: RADIAL SOLUTIONS ON FLRW FOR $k = +1$ AND $k = -1$

We consider the FLRW metric in comoving coordinates $(t, \chi, \theta, \varphi)$:

$$ds^2 = -dt^2 + a(t)^2 (d\chi^2 + S_k(\chi)^2 d\Omega^2), \quad S_k(\chi) = \begin{cases} \sin \chi, & k = +1, \\ \sinh \chi, & k = -1. \end{cases}$$

The non-relativistic Schrödinger equation (zero potential $V = 0$) is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m a(t)^2} \Delta_h \Psi,$$

where for a radial function $R(\chi)$, the spatial Laplacian is equal to

$$\Delta_h R(\chi) = \frac{1}{S_k(\chi)^2} \frac{d}{d\chi} \left(S_k(\chi)^2 \frac{dR}{d\chi} \right).$$

1) **Case $k = +1$ (sphere S^3).** Let $u(\chi) = \sin \chi R(\chi)$. The eigenequation $\Delta_h R + \lambda R = 0$ becomes

$$u''(\chi) + (\lambda + 1)u(\chi) = 0.$$

The regular solutions on $[0, \pi]$ are

$$u_n(\chi) = \sin(n\chi), \quad n = 1, 2, 3, \dots$$

and the corresponding radial functions are

$$R_n(\chi) = \frac{\sin(n\chi)}{\sin \chi}, \quad \lambda_n = n^2 - 1.$$

The time solution for each mode n is

$$\Psi_n(t, \chi) = \mathcal{N}_n a(t)^{-3/2} \frac{\sin(n\chi)}{\sin \chi} \exp\left(-i \frac{\hbar \lambda_n}{2m} \int^t \frac{dt'}{a(t')^2}\right).$$

2) **Case $k = -1$ (hyperbolic space H^3).** Let $u(\chi) = \sinh \chi R(\chi)$. The eigenequation becomes

$$u''(\chi) + (\lambda - 1)u(\chi) = 0.$$

For a continuous spectrum $\lambda > 1$, the regular solutions at $\chi = 0$ are

$$R_\mu(\chi) = \frac{\sin(\mu\chi)}{\sinh \chi}, \quad \lambda = \mu^2 + 1, \quad \mu \geq 0.$$

The complete time solution is

$$\Psi_\mu(t, \chi) = \mathcal{N}(\mu) a(t)^{-3/2} \frac{\sin(\mu\chi)}{\sinh \chi} \exp\left(-i \frac{\hbar \lambda}{2m} \int^t \frac{dt'}{a(t')^2}\right).$$

These solutions are regular at $\chi = 0$ and represent the analogue of the spherical functions $\frac{\sin(\kappa r)}{r}$ of the flat case $k = 0$.

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RADIAL SOLUTION OF THE MONGE-AMPÈRE EQUATION ON THE COMPLEX GRASSMANNIAN $G_{2,4}\mathbb{C}$

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ABSTRACT. In this paper we determine a radial solution of the Monge-Ampère equation on $G_{2,4}\mathbb{C}$ using a change of symplectic variable which helps us to simplify the writing of the latter then put it in radial form which will result in an ordinary differential equation.

Keywords: Calabi-Yau structure, Monge-Ampère equation, Radial solution.

MSC(2020): 51F05, 51F10, 51K05.

INTRODUCTION

In dimension n , a Monge-Ampère equation is a linear combination of the minors of the Hessian matrix [2]

$$Hess(f) = \left(\frac{\partial^2 f}{\partial q_j \partial q_k} \right)_{j,k=1 \dots n}.$$

Let $\Omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \dots + dz_n \wedge d\bar{z}_n)$ be the canonical Kähler form on $\mathbb{C}^n$ and $\alpha = dz_1 \wedge \dots \wedge dz_n$ be the complex volume form. A special lagrangian submanifold is a real n -submanifold L which is lagrangian with respect to Ω and which satisfies the special condition

$$Im(\alpha)|_L = 0.$$

These submanifolds, introduced by Harvey and Lawson in their famous article *Calibrated Geometries* [4] are minimal submanifolds of $\mathbb{C}^n$, and more generally of Calabi-Yau manifolds, transverse in some sense to complex submanifolds. They have been extensively studied after the construction proposed by Strominger, Yau and Zaslow [6] of mirror partners of Calabi-Yau manifolds based upon an hypothetic special lagrangian fibration.

Some examples have been given by many people. We can cite for example Harvey and Lawson [4], Joyce [5] and Bryant [3].

For every smooth function f on $\mathbb{R}^n$, the graph

$$L_f = \left\{ \left(q + i \frac{\partial f}{\partial q} \right), q \in \mathbb{R}^n \right\}$$

is a lagrangian submanifold of $\mathbb{C}^n$. The special lagrangian condition becomes then a differential equation on f , called the special lagrangian equation: Noting that the Hessian matrix is symmetric, this is due to the formula [2]:

$$Im(\det_{\mathbb{C}}(Id + iA)) = \sum_{k=0}^n (-1)^k \sigma_{2k+1}(A) = 0$$

where A is a symmetric matrix.

$$\sum_{k=0}^n (-1)^k \sigma_{2k+1}(\text{Hess}(f)) = 0$$

For $n = 3$: (see [1])

$$\Delta f - \text{hess}(f) = 0$$

A symplectic Monge-Ampère equation in dimension 3 associated with a non-degenerate Monge-Ampère structure (Ω, ω) can be brought back by a change of variables symplectic to one of the following equations

$$\begin{cases} \text{hess}(f) = 1 \\ \Delta f - \text{hess}(f) = 0 \\ \square f + \text{hess}(f) = 0 \end{cases}$$

is integrable if and only if

$$\begin{cases} d\left(\frac{\omega}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0 \\ q \text{ is flat} \end{cases}$$

in dimension 3:

$$\Delta f - \text{hess}(f) = 0 \Leftrightarrow \omega = dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 - dp_1 \wedge dp_2 \wedge dp_3$$

and for the Hessian equation:

$$\text{hess}(f) = 1 \Leftrightarrow \omega = dp_1 \wedge dp_2 \wedge dp_3 - dq_1 \wedge dq_2 \wedge dq_3$$

The geometric structure on $T^*\mathbb{R}^3$ associated with the Lagrangian special equation

$$\Delta f - \text{hess}(f) = 0$$

is the classic Calabi-Yau structure (q, I, Ω, α) with:

$$\begin{cases} q = -\sum_{j=1}^3 dx_j \cdot dx_j + dy_j \cdot dy_j \\ I = \sum_{j=1}^3 \frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \\ \Omega = \sum_{j=1}^3 dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \end{cases}$$

In the same way, the geometric structure associated with the pseudo special Lagrangian equation

$$\square f + \text{hess}(f) = 0$$

is the Calabi-Yau pseudo structure $(q_-, I_-, \Omega, \alpha)$ with:

$$\begin{cases} q_- = dx_1 \cdot dx_1 - dx_2 \cdot dx_2 + dx_3 \cdot dx_3 + dy_1 \cdot dy_1 - dy_2 \cdot dy_2 + dx_3 \cdot dx_3 \\ I_- = -\frac{\partial}{\partial y_1} \otimes dx_1 + \frac{\partial}{\partial x_1} \otimes dy_1 + \frac{\partial}{\partial y_2} \otimes dx_2 - \frac{\partial}{\partial x_2} \otimes dy_2 - \frac{\partial}{\partial y_3} \otimes dx_3 + \frac{\partial}{\partial x_3} \otimes dy_3 \\ \Omega = \sum_{j=1}^3 dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \end{cases}$$

These two structures are the only associated examples to the elliptic Monge-Ampère equations with constant coefficients, there are only two equations on $\mathbb{R}^6$. Note that in any almost Calabi-Yau structure (q, I, Ω, α) is associated with the elliptical structure $(\Omega, Re(\alpha))$. There is therefore a correspondence between almost (pseudo) Calabi-Yau structures and conformal classes of elliptical Monge-Ampère structures. Concerning the hyperbolic structure this concerns the equation

$$hess(f) = 1$$

we will have the structure $(q, S, \Omega, \alpha, \beta)$ with:

$$\begin{cases} q = \sum_{j=1}^3 dx_j \cdot dy_j \\ S = \sum_{j=1}^3 \frac{\partial}{\partial x_j} \otimes dx_j - \frac{\partial}{\partial y_j} \otimes dy_j \\ \Omega = \sum_{j=1}^3 dx_j \wedge dy_j \\ \alpha = dx_1 \wedge dx_2 \wedge dx_3, \quad \beta = dy_1 \wedge dy_2 \wedge dy_3 \end{cases}$$

1. DEFINITION & CONSTRUCTION OF THE METRIC OF $G_{2,4}\mathbb{C}$

1.1. Parameterization of the group $U(2)$. We use Euler decomposition to parameterize $U(2)$:

$$U = e^{i\alpha} \begin{pmatrix} e^{i\frac{\phi+\psi}{2}} \cos \frac{\theta}{2} & e^{i\frac{\phi-\psi}{2}} \sin \frac{\theta}{2} \\ -e^{-i\frac{\phi-\psi}{2}} \sin \frac{\theta}{2} & e^{-i\frac{\phi+\psi}{2}} \cos \frac{\theta}{2} \end{pmatrix},$$

Or :

$$\theta \in [0, \pi], \quad \phi, \psi \in [0, 2\pi), \quad \alpha \in [0, 2\pi).$$

The Maurer-Cartan forms are:

$$\omega = U^{-1}dU = i\sigma_0 + \sum_{j=1}^3 \sigma_j \tau_j$$

with :

$$\begin{aligned} \tau_j &= \frac{1}{2}(\text{Pauli matrices}), \quad \sigma_0 = d\alpha + \frac{1}{2} \cos \theta d\psi, \\ \sigma_1 &= \frac{1}{2}(d\theta \sin \psi + d\phi \sin \theta \cos \psi), \\ \sigma_2 &= \frac{1}{2}(d\theta \cos \psi - d\phi \sin \theta \sin \psi), \\ \sigma_3 &= \frac{1}{2}(d\psi + \cos \theta d\phi). \end{aligned}$$

1.2. Grassmannian $G_{2,4}\mathbb{C}$. The space is given by:

$$G_{2,4}(\mathbb{C}) = \frac{U(4)}{U(2) \times U(2)}.$$

Writing an element of $U(4)$ in the form of 2×2 blocks:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in M_2(\mathbb{C}),$$

the horizontal submanifold is imposed by:

$$\omega_{\text{hor}} = A^\dagger dB + C^\dagger dD.$$

The Fubini–Study metric is then given by:

$$ds^2 = \text{Tr}(\omega_{\text{hor}}^\dagger \omega_{\text{hor}})$$

1.3. Final result: explicit expression. Using a diagonal symmetric parameterization

$$Z = \begin{pmatrix} r_1 e^{i\chi_1} & 0 \\ 0 & r_2 e^{i\chi_2} \end{pmatrix}, \quad 0 \leq r_j < 1,$$

the σ_j act independently on each column.

We obtain:

$$ds^2 = \sum_{j=1}^2 \frac{dr_j^2 + r_j^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)}{(1 - r_j^2)^2} + \frac{r_j^2}{(1 - r_j^2)^2} (d\chi_j + \sigma_0)^2$$

Where by substituting the explicit forms:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \frac{1}{4} (d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2),$$

$$\sigma_0 = d\alpha + \frac{1}{2} \cos \theta d\psi.$$

This provides an explicit compact expression in spherical coordinates.

1.4. Dimension and Reductions. By fixing the gauge $\alpha = 0$ and $\chi_1 = \chi_2$ (quaternionic structure), we recover the symmetric space metric:

$$G_{2,4}(\mathbb{C}) \simeq \frac{SO(6)}{SO(4) \times SO(2)}.$$

$$\det(g) = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2)^2 (\cos^2 \theta_1 - \cos^2 \theta_2)^2 = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2.$$

2. SOLUTION OF THE FIRST MONGE-AMPÈRE EQUATION ON $G_{2,4}\mathbb{C}$

The complex dimension of $\text{Gr}(k, n)$ is expressed by

$$\dim_{\mathbb{C}} \text{Gr}(k, n) = k(n - k).$$

Consequently, the real dimension is

$$\dim_{\mathbb{R}} \text{Gr}(k, n) = 2k(n - k).$$

Calculation for $\text{Gr}(2, 4)$. Let's apply the formula with $k = 2$ and $n = 4$:

$$\dim_{\mathbb{C}} \text{Gr}(2, 4) = 2 \cdot (4 - 2) = 4,$$

and therefore

$$\dim_{\mathbb{R}} \text{Gr}(2, 4) = 2 \cdot 4 = 8.$$

Note. This formula can be seen by counting the number of degrees of freedom required to choose a k -dimensional subspace: we can represent this by a $k \times (n-k)$ matrix in a local affine map, hence $k(n-k)$ complex coordinates. The equation of the determinant of the Hessian is written as follows:

$$\text{hess}(f) = 1$$

Consider the complex Grassmannian $\text{Gr}(2, 4) = U(4)/(U(2) \times U(2))$ with an invariant metric given by the Maurer-Cartan forms $\sigma_i^{(U)}, \sigma_i^{(V)}$ and the spherical angles θ_1, θ_2 .

The radial distance is defined by:

$$r = \sqrt{\theta_1^2 + \theta_2^2}.$$

The radial Hessian Hess_r is decomposed into blocks:

$$\text{Hess}_r = \begin{pmatrix} 0 & 0 \\ 0 & H_{\text{angular}} \end{pmatrix},$$

where the angular block corresponds to the directions orthogonal to the radial direction.

Note. - The angular factor comes from the directions on $U(2) \times U(2)$. - The pure radial direction has a zero eigenvalue, hence the use of the pseudo-determinant. - This factorization is useful for radial integrals and calculating the volume on $\text{Gr}(2, 4)$.

Hessian on $\text{Gr}(2, 4)$. Let $u : \text{Gr}(2, 4) \rightarrow \mathbb{R}$ be a smooth function depending only on the radial angles θ_1, θ_2 of the Grassmannian.

The invariant metric is decomposed into a radial block and an angular block:

$$g = d\theta_1^2 + d\theta_2^2 + g_{\text{angular}},$$

where g_{angular} is the metric over the factor $U(2) \times U(2)$.

The Hessian of u in local coordinates $(\theta_1, \theta_2, \psi_U, \vartheta_U, \phi_U, \psi_V, \vartheta_V, \phi_V)$ is decomposed into blocks:

$$\text{Hess}(u) = \begin{pmatrix} H_{\text{radial}} & 0 \\ 0 & H_{\text{angular}} \end{pmatrix},$$

with

$$H_{\text{radial}} = \begin{pmatrix} \frac{\partial^2 u}{\partial \theta_1^2} & \frac{\partial^2 u}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 u}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 u}{\partial \theta_2^2} \end{pmatrix}, \quad H_{\text{angular}} \text{ corresponding to the directions on } U(2) \times U(2).$$

The total determinant is then factored:

$$\det \text{Hess}(u) = \det(H_{\text{radial}}) \cdot \det(H_{\text{angular}}).$$

For the angular directions, we have the explicit factorization:

$$\det(H_{\text{angular}}) = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2,$$

where $C \approx 12.7096$ for the standard normalization of $su(2)$ generators and the gauge $U = V = I, \Delta = 0$.

Thus, for a radial function $u(\theta_1, \theta_2)$:

$$\det \text{Hess}(u) = \det \begin{pmatrix} \frac{\partial^2 u}{\partial \theta_1 \partial \theta_1} & \frac{\partial^2 u}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 u}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 u}{\partial \theta_2 \partial \theta_2} \end{pmatrix} \cdot C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2$$

Let $u : \text{Gr}(2, 4) \rightarrow \mathbb{R}$ be a radial function, that is, $u = u(\theta_1, \theta_2)$, where (θ_1, θ_2) are the principal angles of the Grassmannian.

The radial Hessian is:

$$H_{\text{radial}} = \begin{pmatrix} \frac{\partial^2 u}{\partial \theta_1^2} & \frac{\partial^2 u}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 u}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 u}{\partial \theta_2^2} \end{pmatrix}.$$

The angular block contributes by:

$$\det(H_{\text{angular}}) = C \left(\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2) \right)^2,$$

with $C \approx 12.7096$ for standard normalization.

The Monge-Ampère equation then becomes:

$$\det \text{Hess}(u) = \det(H_{\text{radial}}) \cdot \det(H_{\text{angular}}) = 1,$$

or explicitly:

$$\frac{\partial^2 u}{\partial \theta_1^2} \frac{\partial^2 u}{\partial \theta_2^2} - \left(\frac{\partial^2 u}{\partial \theta_1 \partial \theta_2} \right)^2 = \frac{1}{C \left(\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2) \right)^2}$$

Remarks.

- This is a second-order nonlinear partial differential equation.
- To solve analytically, we can attempt a separation of variables:

$$u(\theta_1, \theta_2) = f(\theta_1) + g(\theta_2),$$

which reduces the equation to two ordinary differential equations.

- For general solutions, we must resort to numerical methods adapted to the Monge-Ampère equations.

Solution radiale approximative. We are looking for a radial solution $u(\theta_1, \theta_2)$ with separation of variables:

$$u(\theta_1, \theta_2) = f(\theta_1) + g(\theta_2).$$

For small angles ($\theta_1, \theta_2 \ll 1$), we approximate:

$$\sin \theta_i \approx \theta_i, \quad \cos \theta_i \approx 1 - \frac{\theta_i^2}{2}, \quad \cos^2 \theta_1 - \cos^2 \theta_2 \approx \theta_2^2 - \theta_1^2.$$

The Monge-Ampère equation then separates into two ordinary differential equations:

$$f''(\theta_1) \sim \frac{1}{\theta_1(\theta_0^2 - \theta_1^2)}, \quad g''(\theta_2) \sim \frac{1}{\theta_2(\theta_2^2 - \theta_0^2)},$$

where θ_0 is an adjustment constant.

By integrating twice, we obtain an explicit approximate solution:

$$u(\theta_1, \theta_2) \approx \frac{1}{\theta_0^2} \left[\theta_1 \log \left| \frac{\theta_1}{\theta_0 - \theta_1} \right| - \theta_1 \right] + \frac{1}{\theta_0^2} \left[\theta_2 \log \left| \frac{\theta_2}{\theta_0 - \theta_2} \right| - \theta_2 \right] + \text{const.}$$

This solution:

- is factorizable and radial,
- approximately satisfies $\det \text{Hess}(u) = 1$ for small angles,
- can serve as an initial condition for a numerical solution on the whole of $\text{Gr}(2, 4)$.

ASYMPTOTIC BEHAVIOR OF THE SOLUTION

For a radial function $u(\theta_1, \theta_2)$ on the complex Grassmannian $\text{Gr}(2, 4)$, the Monge-Ampère equation is written:

$$\det \text{Hess}(u) = u_{\theta_1 \theta_1} u_{\theta_2 \theta_2} - (u_{\theta_1 \theta_2})^2 = \frac{1}{C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2},$$

where $C \approx 12.7096$ is a normalization constant.

Asymptotic behavior near the origin. For $\theta_1, \theta_2 \rightarrow 0$:

$$\sin \theta_i \sim \theta_i, \quad \cos \theta_i \sim 1, \quad \cos^2 \theta_1 - \cos^2 \theta_2 \sim \theta_2^2 - \theta_1^2.$$

Assuming an additive factorizable solution $u(\theta_1, \theta_2) = f(\theta_1) + g(\theta_2)$, we obtain:

$$f''(\theta_1) \sim \frac{1}{\theta_1(\theta_0^2 - \theta_1^2)}, \quad g''(\theta_2) \sim \frac{1}{\theta_2(\theta_2^2 - \theta_0^2)}.$$

After integrating twice, the asymptotic behavior near the origin is:

$$u(\theta_1, \theta_2) \sim \theta_1 \log \theta_1 + \theta_2 \log \theta_2 + \text{const.}$$

Asymptotic behavior near the boundary. For $\theta_1, \theta_2 \rightarrow \pi/2$:

$$\cos \theta_i \sim \pi/2 - \theta_i, \quad \sin \theta_i \sim 1, \quad \cos^2 \theta_1 - \cos^2 \theta_2 \sim (\pi/2 - \theta_2)^2 - (\pi/2 - \theta_1)^2.$$

The angular factor tends to zero, and to maintain $\det \text{Hess}(u) = 1$, the second derivatives must diverge:

$$u_{\theta_i \theta_i} \sim \frac{1}{\pi/2 - \theta_i}.$$

After integration, the asymptotic behavior near the boundary is:

$$u(\theta_1, \theta_2) \sim -(\pi/2 - \theta_1) \log(\pi/2 - \theta_1) - (\pi/2 - \theta_2) \log(\pi/2 - \theta_2) + \text{const.}$$

Summary.

$$\begin{cases} \theta_i \rightarrow 0 : & u(\theta_1, \theta_2) \sim \theta_1 \log \theta_1 + \theta_2 \log \theta_2, \\ \theta_i \rightarrow \pi/2 : & u(\theta_1, \theta_2) \sim -(\pi/2 - \theta_1) \log(\pi/2 - \theta_1) - (\pi/2 - \theta_2) \log(\pi/2 - \theta_2). \end{cases}$$

- Near the origin, the solution increases smoothly with a positive logarithmic term. - Near the edges of the principal angle space, the solution diverges weakly in a negative logarithmic manner to compensate for the collapsing metric. - Between these two regimes, the solution is smooth and monotonous.

Let $\varphi = \sqrt{\theta_1^2 + \theta_2^2}$

$$\begin{aligned} u_{\theta_i} &= u' \frac{\theta_i}{\varphi} \\ u_{\theta_i \theta_j} &= u'' \frac{\theta_i \theta_j}{\varphi^2} + u' \left(\frac{\delta_{ij}}{\varphi} - \frac{\theta_i \theta_j}{\varphi^3} \right) = \delta_{ij} \frac{u'}{\varphi} + \left(\frac{u''}{\varphi^2} - \frac{u'}{\varphi^3} \right) \theta_i \theta_j \\ \frac{\text{hess}(u)}{\det(g)} &= \left(\frac{u'}{\varphi} \right) \det[\delta_{ij} + \frac{\varphi}{u'} \left(u'' - \frac{u'}{\varphi} \right) \frac{\theta_i \theta_j}{\varphi^2}] = \left(\frac{u'}{\varphi} \right) \left[1 + \frac{\varphi}{u'} \left(u'' - \frac{u'}{\varphi} \right) \right] = \left(\frac{u'}{\varphi} \right) u'' \end{aligned}$$

We introduce the radial coordinate

$$\varphi = \sqrt{\theta_1^2 + \theta_2^2}, \quad \theta_1 = \varphi \cos \alpha, \quad \theta_2 = \varphi \sin \alpha, \quad \alpha \in [0, \pi/2].$$

Then the differentials are written:

$$d\theta_1^2 + d\theta_2^2 = d\varphi^2 + \varphi^2 d\alpha^2.$$

Thus, the radial block of the metric is diagonal with:

$$\det(g_{\text{radial}}) = \varphi^2.$$

Angular block of the metric. The angular block, derived from the Maurer-Cartan forms of $U(4)$, has the following determinant (in coordinates (θ_1, θ_2)):

$$\det(g_{\text{ang}}) = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2,$$

where C is a normalization constant.

Passing according to φ . For $\varphi \ll 1$, we can approximate:

$$\sin \theta_1 \approx \theta_1 = \varphi \cos \alpha, \quad \sin \theta_2 \approx \theta_2 = \varphi \sin \alpha,$$

$$\cos \theta_1 \approx 1, \quad \cos \theta_2 \approx 1, \quad \cos^2 \theta_1 - \cos^2 \theta_2 \approx \theta_2^2 - \theta_1^2 = \varphi^2 (\sin^2 \alpha - \cos^2 \alpha).$$

So :

$$\det(g_{\text{ang}}) \sim C (\varphi \cos \alpha \cdot \varphi \sin \alpha \cdot 1 \cdot 1 \cdot \varphi^2 (\sin^2 \alpha - \cos^2 \alpha))^2 = C \varphi^8 (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2.$$

Complete determinant of the metric. The complete determinant of the metric is the product of the radial block and the angular block:

$$\det(g) = \det(g_{\text{radial}}) \cdot \det(g_{\text{ang}}) \sim C \varphi^2 \cdot \varphi^8 (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2,$$

or finally:

$$\boxed{\det(g) \sim C \varphi^{10} (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2, \quad \varphi = \sqrt{\theta_1^2 + \theta_2^2}.}$$

Notes:

- For $\varphi \ll 1$, the radial block determinant is dominated by φ^2 and the angular block contributes by φ^8 , giving $\det(g) \sim \varphi^{10}$.
- The angular factor $(\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2$ encodes the directional dependence of the metric.
- This expression is useful for volume calculations, Monge-Ampère-type integrals, or Laplacian calculations in radial coordinates.

Consider the radial equation for $u(\varphi)$:

$$\frac{u'(\varphi)}{\varphi} u''(\varphi) C \varphi^{10} (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2 = 1,$$

where C is a constant and α is a fixed angular parameter.

Formatting. We define

$$K(\alpha) := (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2.$$

The equation then becomes

$$u' u'' C K(\alpha) \varphi^9 = 1 \implies u' u'' = \frac{1}{C K(\alpha) \varphi^9}.$$

Substitution $v(\varphi) = u'(\varphi)$. Let $v(\varphi) = u'(\varphi)$, then $v' = u''$. The equation becomes

$$vv' = \frac{1}{CK(\alpha)\varphi^9}.$$

Separation of variables:

$$v dv = \frac{1}{CK(\alpha)}\varphi^{-9} d\varphi.$$

Integration. - Left side: $\int v dv = \frac{v^2}{2}$ - Right side: $\int \varphi^{-9} d\varphi = -\frac{1}{8}\varphi^{-8}$

We therefore obtain:

$$\frac{v^2}{2} = -\frac{1}{8CK(\alpha)}\varphi^{-8} + A,$$

where A is a constant of integration.

Thus:

$$v(\varphi) = u'(\varphi) = \sqrt{2A - \frac{1}{4CK(\alpha)}\varphi^{-8}}.$$

Final Integration for $u(\varphi)$. The radial solution is written:

$$u(\varphi) = \int \sqrt{2A - \frac{1}{4CK(\alpha)}\varphi^{-8}} d\varphi, \quad K(\alpha) = (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2.$$

Asymptotic behavior.

- For $\varphi \rightarrow 0$:

$$u(\varphi) \sim \int \frac{1}{\sqrt{4CK(\alpha)}}\varphi^{-4} d\varphi = -\frac{1}{3\sqrt{4CK(\alpha)}}\varphi^{-3}.$$

- For $\varphi \rightarrow \infty$:

$$u(\varphi) \sim \int \sqrt{2A} d\varphi = \sqrt{2A}\varphi.$$

Remarks. - The solution is radial and depends on the angular factor $K(\alpha)$. - The integral for $u(\varphi)$ does not generally admit an elementary antiderivative, but its asymptotic behavior is explicit. - This form is sufficient for numerical or analytical studies of the equation on $\text{Gr}(2, 4)$.

3. SOLUTION OF THE SECOND EQUATION $\Delta f - \text{hess}(f) = 0$ ON $G_{2,4}\mathbb{C}$

Let $u : \text{Gr}(2, 4) \rightarrow \mathbb{R}$ be a smooth function and g be the invariant Riemannian metric on $\text{Gr}(2, 4)$. The Beltrami Laplacian is defined by

$$\Delta_g u = \text{div}_g(\text{grad}_g u) = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^d \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial u}{\partial x^j} \right),$$

where $d = \dim_{\mathbb{R}} \text{Gr}(2, 4) = 8$ and g^{ij} is the inverse of the metric matrix g_{ij} .

Case of a radial function. If u depends only on the radial angles (θ_1, θ_2) , the metric is decomposed into a radial block and an angular block:

$$g = g_{\text{radial}} + g_{\text{angular}}, \quad g_{\text{radial}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det g_{\text{angular}} = C(\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2.$$

The Laplacian is reduced to the radial block:

$$\Delta_g u(\theta_1, \theta_2) = \frac{1}{\sqrt{\det g_{\text{angular}}}} \sum_{i,j=1}^2 \frac{\partial}{\partial \theta_i} \left(\sqrt{\det g_{\text{angular}}} g_{\text{radial}}^{ij} \frac{\partial u}{\partial \theta_j} \right).$$

Explicit formula. With $C \approx 12.7096$ for standard normalization:

$$\Delta_g u(\theta_1, \theta_2) = \frac{\sum_{i,j=1}^2 \frac{\partial}{\partial \theta_i} \left(\sqrt{C} \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2) g_{\text{radial}}^{ij} \frac{\partial u}{\partial \theta_j} \right)}{\sqrt{C} \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2)}$$

Consider the equation on the complex Grassmannian $\text{Gr}(2, 4)$:

$$\Delta f - \text{Hess}(f) = 0,$$

where Δf is the Laplacian associated with the Riemannian metric of the Grassmannian and $\text{Hess}(f)$ is the Hessian of f .

$\Delta f - \text{hess}(f) = 0 \Leftrightarrow \Delta f = \text{hess}(f)$, let's look for a radial solution of this equation, we write $f(\theta_1, \theta_2) = u(\varphi)$ with $\varphi = \sqrt{\theta_1^2 + \theta_2^2}$. We have: $\frac{\partial f}{\partial \theta_i} = \frac{\theta_i}{r} u'$ and therefore $\frac{\partial^2 f}{\partial \theta_i^2} = \frac{\theta_i^2}{r^2} u'' + \frac{\varphi^2 - \theta_i^2}{\varphi^3} u'$ then,

$$\Delta f = \sum_{k=1}^2 \left(\frac{\theta_k^2}{\varphi^2} u'' + \frac{\varphi^2 - \theta_k^2}{\varphi^3} u' \right) = u'' + \frac{1}{\varphi} u' = \frac{u'(\varphi)}{\varphi} u''(\varphi) C \varphi^{10} (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2$$

$$u'' + \frac{1}{\varphi} u' = \frac{u'}{\varphi} u'' C \varphi^{10} (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2,$$

where C is a constant and α is a fixed angular parameter.

Formatting. We define

$$K(\alpha) := (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2.$$

The equation becomes:

$$u'' + \frac{1}{\varphi} u' - CK(\alpha) \varphi^9 u' u'' = 0.$$

Factorization of u'' :

$$u'' (1 - CK(\alpha) \varphi^9 u') = -\frac{u'}{\varphi}.$$

Substitution $v(\varphi) = u'(\varphi)$. Let $v = u'$, then $v' = u''$. The equation can be rewritten as follows:

$$v' (1 - CK(\alpha) \varphi^9 v) = -\frac{v}{\varphi} \implies \frac{dv}{d\varphi} = -\frac{v}{\varphi (1 - CK(\alpha) \varphi^9 v)}.$$

Implicit form of $v(\varphi)$. The implicit solution is given by the integral:

$$\int (1 - CK(\alpha) \varphi^9 v(\varphi)) \frac{dv}{v} = - \int \frac{d\varphi}{\varphi}.$$

This equation defines $v(\varphi)$ implicitly and can be solved numerically.

Final Solution for $u(\varphi)$. Once $v(\varphi)$ is determined, the solution to the initial equation is obtained by:

$$u(\varphi) = \int v(\varphi) d\varphi.$$

Remarks.

- The angular dependence is encapsulated in $K(\alpha)$.
- The equation does not have a simple analytical solution as an elementary function, but the implicit form is sufficient for numerical or asymptotic studies.
- This formalism is directly related to the radial structure of the metric on $\text{Gr}(2, 4)$ and to the determinant of the Hessian.

Form for the derivative $v(\varphi) = u'(\varphi)$. The equation is rewritten:

$$v'(1 - CK(\alpha)\varphi^9 v) = -\frac{v}{\varphi}, \quad v = u'.$$

Asymptotic solution for $\varphi \rightarrow 0$. For $\varphi \ll 1$, we have $CK(\alpha)\varphi^9 v \ll 1$, therefore:

$$v' \approx -\frac{v}{\varphi} \implies v(\varphi) \sim \frac{A}{\varphi}, \quad A \in \mathbb{R}.$$

Integration for $u(\varphi)$:

$$u(\varphi) \sim A \ln \varphi + B, \quad \varphi \rightarrow 0,$$

where B is a constant of integration.

Asymptotic solution for $\varphi \rightarrow \infty$. For $\varphi \gg 1$, the nonlinear factor dominates and we have:

$$v(\varphi) = u'(\varphi) \sim \frac{1}{CK(\alpha)\varphi^9}.$$

Integration for $u(\varphi)$:

$$u(\varphi) \sim -\frac{1}{8CK(\alpha)}\varphi^{-8} + \text{constant}, \quad \varphi \rightarrow \infty.$$

Remarks.

- The complete elementary function solution does not exist due to nonlinearity.
- These expressions give the exact asymptotic behavior for the radial function on $\text{Gr}(2, 4)$.
- The angular dependence is encapsulated in $K(\alpha)$.

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A Visual Exploration of the Inequality $l^e < e^l$: Enhancing Student Understanding Through Geometric Reasoning

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Abstract

This article presents a geometric proof of the classical inequality $l^e < e^l$ for $0 < l \neq e$, designed specifically for mathematics education contexts. Building upon existing visual proofs in the literature, we generalize the approach to encompass all positive real numbers and demonstrate how this geometric argument can enhance student understanding of logarithmic functions, concavity, and inequality reasoning. The proof requires only elementary knowledge of logarithms and coordinate geometry, making it accessible to undergraduate students while providing valuable visual intuition that complements traditional analytic approaches. We provide detailed classroom implementation strategies, discuss connections to important mathematical constants, and situate this work within the broader literature on visual proofs in mathematics education.

Keywords: Visual proof, Mathematics education, Natural logarithm, Tangent line, Pedagogical approach.

1. Introduction

Visual proofs have long been recognized in mathematics education for their ability to make abstract concepts more accessible and intuitively understandable (Nelsen, 1993, 2015). The inequality $l^e < e^l$ for positive real numbers $l \neq e$ is a classical result with applications across various mathematical domains, yet its standard analytic proofs often rely on calculus techniques that may obscure the underlying geometric relationships.

This article presents a unified geometric proof that covers the entire domain $l > 0$, $l \neq e$. Beyond merely proving the inequality, we focus on the pedagogical implications of this approach, demonstrating how it can help students develop deeper understanding of several key mathematical concepts:

- The geometric properties of logarithmic functions
- The relationship between functions and their tangent lines
- The concept of concavity and its algebraic implications
- Strategic use of visual reasoning in analytic contexts

This approach differs from previous works studied by Stylianou (2002) and Presmeg (2006). With its generalization and the explicit connections, we draw to mathematics education research on visualization and proof comprehension.

2. Pedagogical Context and Mathematical Background

Before presenting our geometric proof, it is valuable to situate this work within both the mathematical literature and mathematics education research. The inequality $l^e < e^l$ has been proven through various methods:

2.1 Analytic Approaches

The standard analytic proof uses calculus, examining the function $f(x) = \frac{\ln x}{x}$. Since $f'(x) = \frac{1 - \ln x}{x^2}$, the function has a maximum at $x = e$, yielding $f(l) < f(e)$ for $l \neq e$, which simplifies to $l^e < e^l$. While elegant, this proof requires understanding of derivatives and optimization, concepts that may not be immediately accessible to all students.

2.2 Visual Approaches in Literature

Chakraborty's (2024) visual proof considers only $b > e$ and uses an area comparison argument. Mukherjee and Chakraborty's (2019) proof specifically for $\pi^e < e^\pi$ employs a tangent line argument similar in spirit to ours but limited to that particular case. Both approaches demonstrate the pedagogical value of visualization, but leave room for a more comprehensive treatment of a complete geometric proof valid for all positive $l \neq e$.

2.3 Educational Value of Visual Proofs

Research in mathematics education indicates that visual proofs can serve multiple pedagogical purposes:

- They provide alternative pathways to understanding for visual learners
- They help students develop intuition before formal reasoning
- They offer memorable representations that aid long-term retention
- They encourage exploration and discovery-based learning

Our geometric proof leverages these benefits while addressing a mathematically significant inequality that appears in various advanced contexts, from information theory to mathematical economics.

3. A Unified Geometric Proof

The foundation of our proof lies in the geometric properties of the natural logarithm function $y = \ln x$:

- The function is concave down on its entire domain $(0, \infty)$
- The tangent line at $x = 1$ is $y = x - 1$
- By concavity, $\ln x \leq x - 1$ for all $x > 0$, with equality only at $x = 1$

These properties are visually evident in Figure 1, which forms the basis for our geometric reasoning.

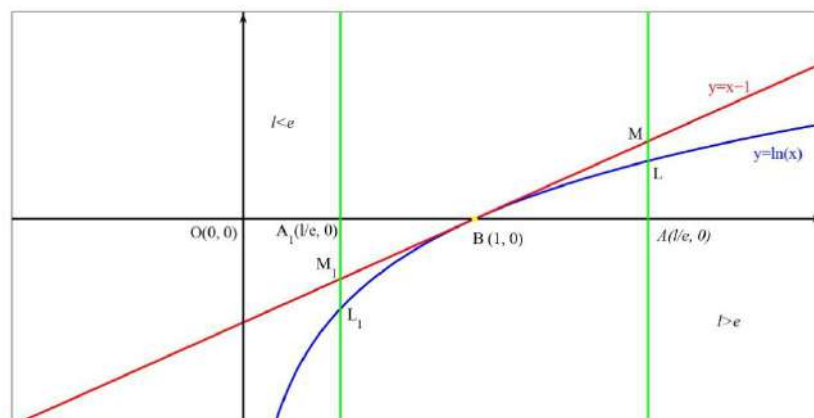


Figure-1: Geometric representation of the natural logarithm function and its tangent line at (1, 0)

Theorem 1. For any positive real number $l \neq e$, the inequality $l^e < e^l$ holds.

Proof. We present a case analysis that covers all positive $l \neq e$:

Case 1: $0 < l < e$

Consider the point $x = l/e$. From Figure 1, we observe geometrically that the vertical distance from the point $(1,0)$ to the curve $y = \ln x$ at $x = l/e$ (point L_1) is greater than the vertical distance to the tangent line $y = x - 1$ at the same x -coordinate (point M_1). Algebraically, this gives:

$$\left| \ln\left(\frac{l}{e}\right) \right| = \overline{A_1 L_1} > \overline{A_1 M_1} = \left| \left(\frac{l}{e} - 1\right) \right|$$

Since $l/e < 1$, both $\ln(l/e)$ and $(l/e - 1)$ are negative, so the absolute values reverse the inequality:

$$-\ln\left(\frac{l}{e}\right) > -\left(\frac{l}{e} - 1\right) \Rightarrow \ln\left(\frac{e}{l}\right) > 1 - \frac{l}{e}$$

Exponentiating both sides (using the monotonicity of the exponential function):

$$\frac{e}{l} > e^{1-\frac{l}{e}} = \frac{e}{e^{\frac{l}{e}}}$$

Cross-multiplying yields:

$$e \cdot e^{\frac{l}{e}} > e \cdot l \Rightarrow e^{\frac{l}{e}} > l \Rightarrow e^l > l^e$$

Case 2: $l > e$

When $l > e$, we have $l/e > 1$. Geometrically, the vertical distance from $(1,0)$ to the curve at $x = l/e$ (point L) is now less than the distance to the tangent line at the same x -coordinate (point M):

$$\ln\left(\frac{l}{e}\right) = \overline{AL} < \overline{AM} = \left(\frac{l}{e} - 1\right)$$

This inequality simplifies to:

$$\ln l - \ln e < \frac{l}{e} - 1 \Rightarrow \ln l - 1 < \frac{l}{e} - 1 \Rightarrow \ln l < \frac{l}{e}$$

Multiplying by $e > 0$:

$$e \ln l < l \Rightarrow \ln l^e < l \Rightarrow l^e < e^l$$

Combining both cases, we get the theorem for all $l > 0, l \neq e$.

4. Educational Extensions and Classroom Applications

The geometric proof presented above offers numerous opportunities for classroom exploration and extension. Below we outline several pedagogical approaches.

4.1 Comparative Analysis of Proof Techniques

Instructors can facilitate rich classroom discussions by comparing different proof strategies:

- **Analytic approach:** Using derivatives to maximize $f(x) = \frac{\ln x}{x}$
- **Visual approach presented here:** Geometric comparison of distances
- **Chakraborty's area method:** For the special case $b > e$

Students can discuss the relative strengths of each approach, considering factors like generality, accessibility, elegance, and memorability.

4.2 Exploration with Specific Constants

The theorem yields immediate corollaries for important mathematical constants:

Corollary 2 (Comparison with π). Since $\pi \approx 3.14159 \neq e \approx 2.71828$, we have $\pi^e < e^\pi$.

Corollary 3 (Comparison with the Golden Ratio). For $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803 \neq e$, we have $\phi^e < e^\phi$.

Corollary 4 (Comparison with the Euler-Mascheroni Constant). For $\gamma \approx 0.57721 \neq e$, we have $\gamma^e < e^\gamma$.

These specific cases provide concrete examples that students can verify numerically, building confidence in the general result.

4.3 Interactive Classroom Activities

1. **Visual Exploration:** Using graphing software, students can manipulate the value of l and observe how the distances $\overline{A_1L}$ and $\overline{A_1M}$ change, building intuition before formal proof.
1. **Proof Reconstruction:** After seeing the complete proof, students can attempt to reconstruct it from the diagram alone, strengthening their visual reasoning skills.
2. **Extension Questions:**
 - What happens exactly at $l = e$? Why is this case excluded?
 - How would the proof change if we used a tangent line at a point other than $x = 1$?
 - Can you develop a similar geometric proof for other inequalities involving exponential and logarithmic functions?

5. Pedagogical Implications

The geometric proof of $l^e < e^l$ presented in this article offers several advantages for mathematics education:

- The proof requires only elementary knowledge of logarithms, exponents, and coordinate geometry, making it accessible to a wide range of students, including those in pre-calculus or early calculus courses.
- By linking algebraic inequalities to geometric distances, the approach provides visual reinforcement that can aid comprehension and retention, particularly for visual learners.
- Unlike previous visual proofs that treated special cases, our approach handles all positive $l \neq e$ in a unified framework, demonstrating the power of geometric reasoning to establish general results.
- Presenting this visual proof alongside traditional analytic approaches encourages students to appreciate multiple problem-solving strategies and develops flexible mathematical thinking.

We encourage instructors to incorporate this geometric approach into their teaching of logarithmic and exponential functions, either as a primary explanation or as a supplementary perspective that enriches students' understanding. The visual nature of the proof makes it particularly suitable for interactive classroom presentations, and the accompanying diagram can serve as a lasting visual reference for students.

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Promoting Risk-Taking in Mathematics Classrooms through Emotionally Safe Learning Environments: A Narrative Review of Literature

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Abstract

In many Indian school classrooms, mathematics teaching still leans on procedure, is driven by examinations, and offers students few chances to try out different solution paths. These conditions can push learners away from intellectual risk-taking, feeding a fear of mistakes, reluctance to voice mathematical ideas, and lower engagement with problem-solving. This paper offers a narrative review of Indian and international scholarship on academic risk-taking in mathematics, focusing on the part played by emotionally safe learning environments. Drawing on constructivist, sociocultural, and achievement-goal perspectives, it brings together work on teacher facilitation, classroom discourse, error-friendly pedagogy, productive struggle, mathematics anxiety, and growth mindset to explain when students are willing, or unwilling, to attempt unfamiliar mathematical tasks. Sources were located through Google Scholar, ERIC, and Scopus and grouped thematically into six recurring categories that shape the structure of the review. The review reads this international evidence against the Indian policy backdrop set out in the National Curriculum Framework 2005 and the National Curriculum Framework for School Education 2023, pointing to a persistent gap between these curricular aims and everyday classroom practice, as well as a shortage of classroom-based research within Indian mathematics education. It proposes a conceptual framework connecting classroom conditions, emotional safety, psychological factors, and academic risk-taking, and closes by drawing out implications for teacher facilitation, assessment design, and teacher professional development as directions for future classroom-based research across diverse Indian school settings.

Keywords: Academic Risk-taking; Mathematics Anxiety; Emotionally Safe Classrooms; Growth Mindset; Indian School Education

1. Introduction

For many students, mathematics is bound up with fear, anxiety, and avoidance, especially in systems that place heavy weight on examination results [2]. Yet mathematics is fundamentally a discipline of reasoning, exploration, and creative problem-solving; when learners instead experience it mainly as a set of fixed procedures with single correct answers, this shapes how they see themselves as mathematical thinkers and can make them reluctant to engage with tasks that involve uncertainty or exploration. Within mathematics education, academic risk-taking describes students' readiness to tackle challenging problems, try out alternative strategies, voice half-formed ideas, ask questions, and learn from mistakes without fearing judgement. Recent conceptual work frames academic risk-taking in terms of

affective, preferred-difficulty, and action dimensions, linking it empirically to mastery-goal orientation and need for cognition [1]. Genuine mathematical learning asks students to go beyond routine procedure and into conjecturing, reasoning, justifying, and revising their own thinking [4] – something they are far more willing to do when the classroom feels supportive, respectful, and emotionally safe.

Achievement goal theory holds that classrooms built around learning, improvement, and understanding foster deeper engagement, while classrooms dominated by comparison and fear of failure make students less willing to take part [2]. A related strand of work on productive struggle shows that grappling with difficult mathematical ideas, without an immediate resolution, is itself central to building conceptual understanding: analysing 186 student–teacher interaction episodes in real classrooms, Warshauer found that whether teachers removed the challenge for struggling students or supported them through it determined whether the struggle turned productive or merely frustrating [16]. In a similar vein, Kapur's research on productive failure found that students who first wrestled with novel problems on their own – even when they did not succeed – went on to develop a deeper conceptual understanding than peers who received direct instruction from the start [9].

International scholarship consistently points to the importance of emotionally safe mathematics classrooms. Boaler [4] contends that students build stronger mathematical mindsets in classrooms that reward exploration, multiple solution approaches, and learning from mistakes, rather than classrooms narrowly focused on speed and correctness. Work on growth mindset suggests that what students believe about their own mathematical ability shapes how far they will persist through difficulty [5]. Surveying 159 university students, Hübner and Pfost found that beliefs about errors – seeing them as threatening versus informative – predicted students' willingness to take academic risks even after accounting for general achievement motivation [8]; this points to error beliefs as a distinct, teachable target for intervention rather than a mere by-product of ability or confidence.

By contrast, classrooms that treat wrong answers negatively, or teachers who prize procedural accuracy above everything else, can unintentionally breed a fear of failure that keeps students from voicing their reasoning or attempting unfamiliar problems. Both the National Curriculum Framework 2005 and the National Curriculum Framework for School Education 2023 likewise stress that good mathematics teaching should build reasoning, problem-solving, and meaningful engagement, not procedural fluency alone [11,12].

Through these frameworks, the Indian school system has acknowledged the need for learner-centred, inquiry-based approaches, calling for conceptual understanding, joyful learning, and classrooms where students take an active part. In practice, however, mathematics learning in many schools remains shaped by examination-oriented teaching, teacher-centred instruction, and few openings for open-ended mathematical discussion – conditions that plausibly feed mathematics anxiety and a reluctance to take on unfamiliar tasks.

Research on mathematics anxiety shows that emotional factors have a marked bearing on students' attitudes, confidence, and achievement [3]. International literature has a good deal to say about mathematical risk-taking, growth mindset, and supportive classroom practice [1,3,4,5,8], yet comparatively little of it looks specifically at how emotionally safe classroom environments shape risk-taking within Indian school

settings. This gap matters because risk-taking is closely bound up with mathematical reasoning, creativity, persistence, and the development of a positive mathematical identity.

This review brings together theoretical perspectives and empirical findings on mathematics anxiety, classroom discourse, teacher facilitation, productive struggle, and growth-oriented teaching practices, setting international evidence alongside the Indian educational context to identify key themes, research gaps, and pedagogical implications for mathematics classrooms.

2. Objectives of the Review

This review is guided by the following objectives:

1. To review existing literature on academic risk-taking in mathematics learning.
2. To examine the role of emotionally safe classroom environments in promoting student participation and confidence.
3. To analyse the influence of psychological factors – mathematics anxiety, achievement goal orientation, and growth mindset – on students' willingness to engage in mathematical risk-taking.
4. To identify gaps in research on risk-taking behaviour in Indian mathematics classrooms.
5. To propose a conceptual framework explaining how emotionally safe classroom environments can promote academic risk-taking in mathematics learning.

3. Methods

This paper takes a narrative, thematic approach to the literature rather than a formal systematic review. This choice reflects the review's purpose: to bring together theoretical perspectives and empirical findings from a body of work on academic risk-taking that is still emerging and conceptually scattered, and to connect that international evidence with the Indian curricular context. That kind of synthesising, interpretive work is better suited to narrative review than to the effect-size aggregation that a systematic review or meta-analysis is designed to produce.

The sources were located through targeted searches of Google Scholar, ERIC, and Scopus, combining terms such as academic risk-taking, mathematical risk-taking, emotionally safe classrooms, mathematics anxiety, growth mindset in mathematics, productive struggle, error-friendly pedagogy, and Indian mathematics education. Preference was given to peer-reviewed empirical and theoretical work in mathematics education and educational psychology, together with foundational texts retained regardless of their publication date because of the continuing influence they carry in the field. NCERT policy documents – the National Curriculum Framework 2005 and the National Curriculum Framework for School Education 2023 – were included to anchor the review in the Indian curricular context, while sources dealing solely with higher-education mathematics, or with general risk-taking unconnected to education, were left out. Being a narrative review, the selection of sources was purposive and thematic rather than exhaustive; it does not claim to offer a full census of the literature, a limitation taken up later in the paper.

Thematic analysis organised the selected literature into six recurring categories: the nature of academic risk-taking in mathematics; emotional safety and classroom climate; mathematics anxiety as a psychological barrier; achievement goal orientation; growth mindset and error-friendly pedagogy; and the Indian curricular and classroom context. These six themes provide the structure for the review that follows.

4. Results

4.1 Academic Risk-Taking in Mathematics Learning

Academic risk-taking in mathematics is students' willingness to engage unfamiliar problems, try out multiple solution strategies, put forward tentative ideas, ask questions, and revise their thinking without dreading a negative verdict. These behaviours matter for meaningful mathematical learning because they support reasoning, creativity, and deeper conceptual understanding [2,4]. Across two validation studies with 236 undergraduate students, Abercrombie and colleagues modelled academic risk-taking along four dimensions – affect, preferred difficulty, and two facets of action – and found it was predicted by mastery-goal orientation and need for cognition rather than by general academic ability [1]. This reinforces the view that risk-taking is best understood as a disposition shaped by classroom goal structures, not as a fixed trait.

Classrooms that emphasise speed, correctness, and single right answers can hold students back from joining mathematical discussion or trying difficult tasks. When success is judged mainly by getting the correct answer, students may read their own mistakes as proof of inability rather than as chances to learn [4,5]. Achievement goal theory suggests that mastery-oriented classrooms nurture persistence, while performance-focused ones can heighten fear of failure and avoidance [2]. Fostering academic risk-taking in mathematics therefore calls for a move away from answer-oriented instruction and toward environments that value questioning, experimentation, and productive error.

4.2 Emotional Safety and Classroom Climate

An emotionally safe mathematics classroom is one where students feel respected and accepted, and confident enough to voice their ideas without fear of embarrassment – a place where errors and half-formed reasoning are treated as a normal part of learning rather than a sign of failure. Vygotsky's social constructivism, from a sociocultural standpoint, holds that understanding grows through interaction, dialogue, and support from more knowledgeable others; in a mathematics classroom, this plays out as discussion and comparison of solution strategies that help build conceptual understanding and reasoning.

Classroom discourse is central to establishing this sense of safety. Teachers who invite students to explain their thinking, give them enough wait time, and respond constructively to wrong answers create conditions where learners are readier to participate and to refine their ideas [10]. Hübner and Pfof's survey research found that university students' beliefs about errors predicted their willingness to take academic risks independently of prior achievement [8], which suggests that how a classroom frames errors is itself a lever for change rather than simply a reflection of students' ability. Where evaluation is quick and correctness is prized above all, students are, by contrast, more likely to keep tentative ideas to themselves.

Teacher feedback carries particular weight in shaping how students see their own mathematical ability. Feedback centred on effort, strategy, and improvement helps students treat challenges as chances to learn, in keeping with growth mindset principles [5]. The National Curriculum Framework 2005 calls for mathematics education to move past rote memorisation and toward conceptual understanding and reasoning [11], and the National Curriculum Framework for School Education 2023 similarly pushes for learner-centred, experiential approaches that build curiosity and active participation

[12]. Despite these recommendations, a gap persists between what the curriculum asks for and what happens in classrooms: examination pressure, time constraints, and conventional teaching methods often leave little room for dialogue and collaborative exploration.

4.3 Mathematics Anxiety as a Barrier to Risk-Taking

Mathematics anxiety – the tension, apprehension, and fear that get in the way of mathematical performance and participation [6] – is a well-documented barrier to risk-taking. Students who experience it tend to steer clear of challenging tasks, hesitate to share their reasoning, and lean on procedural methods that feel safer than exploratory ones. This anxiety is closely tied to a fear of failure and of being judged. When students see mathematics classrooms as places where mistakes are penalised, they may avoid unfamiliar problems to protect their self-confidence, which then limits how much productive struggle and conceptual growth they experience [4, 6]. In Indian schools specifically, examination-oriented learning, competitive academic pressures, and a strong premium on correct answers often intensify mathematics anxiety, echoing the wider literature on how test-related anxiety affects cognition and learning [3]. Large class sizes and few chances for individual expression can further crowd out meaningful mathematical dialogue. Tackling mathematics anxiety therefore takes more than sharpening mathematical skills; it calls for classroom environments in which students feel supported and confident enough to explore mathematical ideas.

4.4 Achievement Goal Orientation and Classroom Practices

Achievement goal theory draws a line between mastery-oriented classrooms – built around learning, understanding, effort, and improvement – and performance-oriented classrooms, which are built around grades, comparison, and demonstrated ability relative to peers [2]. Mastery-oriented settings tend to nurture persistence, curiosity, and a willingness to take on challenging tasks; when students see the point of mathematics learning as building understanding rather than just producing the correct answer, they become more willing to try out alternative strategies and to learn from mistakes [2, 4]. Performance-oriented classrooms, on the other hand, can breed a fear of failure and push students to avoid situations in which their ability might be publicly called into question. Teachers' own practices shape which of these goal structures takes hold: open-ended questioning, encouraging multiple solution methods, collaborative problem-solving, and formative assessment all help build mastery-oriented mathematics classrooms in keeping with constructivist views of learning [7,14].

4.5 Growth Mindset and Error-Friendly Pedagogy

Growth mindset theory maintains that ability can be built through effort, good strategies, and feedback. Dweck [5] proposed that students with a growth mindset are more likely to take on challenges, keep going through difficulty, and see mistakes as chances to improve. Boaler [4] makes a related point: classrooms that reward exploration, multiple approaches, and reflection help students build both flexible mathematical thinking and a stronger sense of themselves as mathematical thinkers. A related idea, error-friendly pedagogy, treats mistakes as a valuable resource for learning rather than a mark of failure. In a classroom-based study of middle-school mathematics, Warshauer found that how teachers responded to student struggle – stepping in to remove the difficulty versus scaffolding students through it while keeping the challenge intact – was what separated productive struggle from plain frustration [16]. Kapur's studies of productive failure reached a similar conclusion: students who

wrestled with novel problems before any instruction, even when their first attempts fell short, went on to outperform students taught the procedure directly, at least on measures of conceptual understanding [9]. Taken together, these findings suggest that allowing struggle and error, and scaffolding students supportively through them, is not incidental to learning mathematics but one of the mechanisms through which deeper understanding is actually built. Because uncertainty and exploration sit at the heart of mathematical problem-solving, easing students' fear of being wrong matters a great deal for building their confidence and persistence. Growth mindset practices and error-friendly pedagogy therefore feed into emotionally safe learning environments, turning classrooms from spaces fixed on correctness into communities that value reasoning and learning through mistakes.

4.6 Indian School Context and Research Gaps

International research has a good deal to say about the value of emotional safety, growth-oriented practices, and supportive classroom environments for mathematical risk-taking, yet comparatively little of it looks at these dimensions within Indian mathematics classrooms. The National Curriculum Framework 2005 and the National Curriculum Framework for School Education 2023 call for learner-centred approaches, conceptual understanding, and classrooms that encourage active participation [11,12], but translating these curricular goals into everyday classroom practice continues to be difficult. Most existing research on mathematics education in India has centred on achievement levels, curriculum effectiveness, and instructional strategy. Far fewer studies look at how classroom emotional climate relates to students' willingness to take mathematical risks, share their reasoning, and take an active part. There is also a need for classroom-based intervention studies and teacher-facing frameworks that translate the growth-mindset and error-friendly principles established internationally into strategies that work in Indian classroom settings – settings that often mean large classes and examination-driven assessment.

Table 1. Thematic Synthesis of Literature on Academic Risk-Taking and Emotionally Safe Mathematics Learning Environments

Theme	Key Focus	Major Insights from Literature	Educational Implications
Academic Risk-Taking in Mathematics	Willingness to attempt unfamiliar problems, explore strategies, and communicate ideas	Risk-taking supports reasoning, creativity, and conceptual understanding; fear of failure restricts participation [1, 4, 7].	Encourage exploratory tasks, multiple solution approaches, and mathematical discussion.
Emotional Safety and Classroom Climate	Respectful interactions, acceptance of errors, supportive norms	Emotionally safe classrooms increase students' confidence to share ideas; teacher responses strongly shape participation [8, 10].	Create error-friendly classrooms where incomplete thinking is welcomed.

Mathematics Anxiety and Psychological Barriers	Fear, stress, negative beliefs, avoidance behaviours	Anxiety reduces willingness to attempt challenging tasks; supportive environments lower anxiety and raise confidence [3, 6].	Use collaborative learning, constructive feedback, and low-threat assessment.
Teacher Facilitation and Classroom Discourse	Questioning patterns, feedback, wait time, discussion practices	Teachers who value student explanations and reasoning promote deeper engagement [10].	Encourage dialogue, reflection, and justification of solutions.
Achievement Goal Orientation	Mastery-oriented versus performance-oriented classroom structures	Mastery-focused classrooms promote persistence; performance pressure increases avoidance [1, 2].	Focus assessment on growth, effort, and improvement rather than comparison.
Growth Mindset and Error-Friendly Pedagogy	Viewing ability as developable; mistakes as learning opportunities	Growth-oriented messages encourage persistence and willingness to engage with difficult tasks [5, 15].	Normalise mistakes and use errors as resources for conceptual development.
Indian Mathematics Education Context	Curriculum reforms, examination culture, classroom realities	Policy frameworks promote learner-centred mathematics education, but implementation challenges remain [11, 12].	Develop teacher resources and classroom interventions suited to Indian contexts.

As Table 1 shows, the thematic synthesis points to academic risk-taking in mathematics as the product of several interconnected factors – classroom emotional climate, teacher practices, students' beliefs about learning, and assessment approaches among them. Taken together, the literature suggests that building emotionally safe mathematics classrooms means shifting away from performance-oriented practice and toward environments that encourage exploration, reasoning, and learning through error.

Table 2. Conceptual Link between Emotional Safety and Mathematical Risk-Taking

Component of Emotional Safety	Classroom Practices	Influence on Students' Mathematical Behaviour	Expected Outcomes
Trust and	Creating respectful	Students become more	Increased

Psychological Security	interactions, encouraging questions, ensuring students feel valued	comfortable expressing incomplete ideas and attempting unfamiliar problems.	participation and confidence.
Acceptance of Errors as Learning Opportunities	Discussing mistakes constructively, analysing incorrect solutions, encouraging reflection	Students develop a positive attitude towards errors and become less afraid of failure.	Greater willingness to experiment and revise thinking.
Supportive Teacher Facilitation	Giving constructive feedback, open-ended questions, adequate wait time	Students engage more actively in mathematical discussion and reasoning.	Stronger mathematical communication and understanding.
Collaborative Classroom Culture	Peer discussion, group problem-solving, sharing multiple strategies	Students learn from diverse perspectives and gain confidence in presenting ideas.	Enhanced collaboration, creativity, and problem-solving.
Mastery-Oriented Learning Environment	Emphasising effort, improvement, reasoning over correct answers	Students view challenges as opportunities for growth rather than threats.	Increased persistence and academic risk-taking.
Growth Mindset and Productive Struggle	Encouraging challenging tasks; reinforcing that ability develops through practice	Students develop resilience and continue working through difficulties.	Deeper mathematical engagement and flexible thinking.
Process-Oriented Assessment	Using formative feedback, reflection, evaluation of reasoning	Students focus on learning and improvement rather than avoiding mistakes.	Reduced mathematics anxiety and stronger mathematical identity.

5. Discussion

5.1 Conceptual Framework

Building on the thematic synthesis above, Figure -1 sets out a conceptual framework linking classroom conditions, emotional safety, and academic risk-taking in mathematics learning. Classroom conditions – teacher facilitation, error-friendly discourse, mastery-goal structures, assessment practices, and the wider Indian curricular context – combine to shape whether a classroom feels emotionally safe. Emotional safety, in turn, is bound up with three interrelated psychological factors:

lower mathematics anxiety, a growth mindset or mastery orientation, and self-efficacy. Together, these factors shape how willing students are to take academic risks – attempting unfamiliar tasks, asking questions, offering tentative reasoning, and learning from errors – a willingness the literature links to stronger conceptual understanding, mathematical reasoning, persistence, and a positive mathematical identity [1,4,5,8,16].

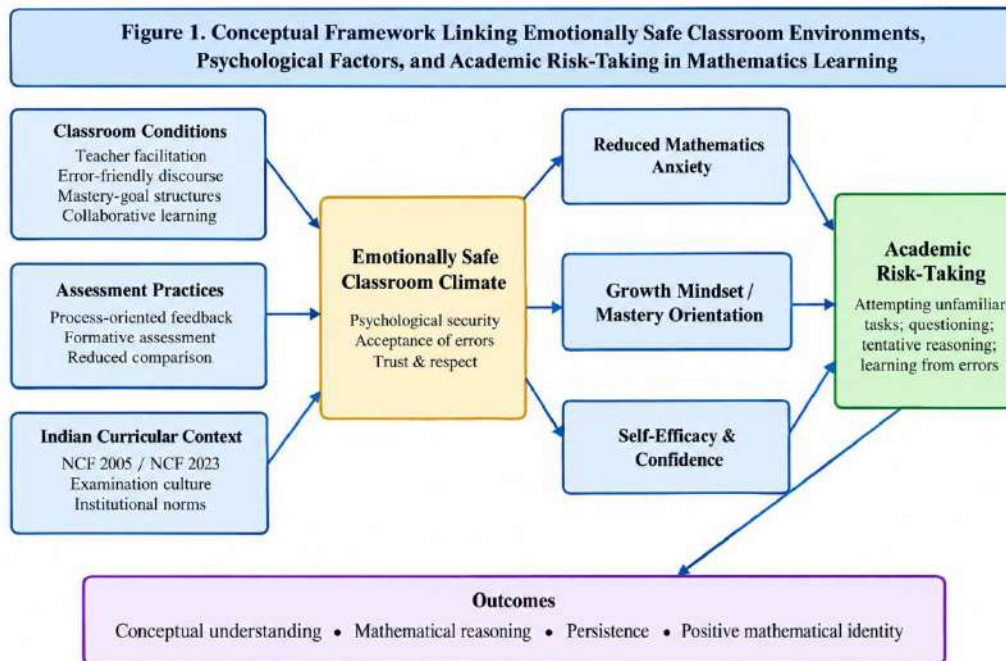


Figure 1. Conceptual framework linking emotionally safe classroom environments to academic risk-taking and mathematical learning outcomes.

5.2 Research Gap

International research has examined academic risk-taking in mathematics quite extensively, particularly in connection with problem-solving, classroom discourse, student motivation, and mathematical identity, and consistently shows that emotionally supportive learning environments lift students' confidence, participation, and willingness to take on challenging mathematical tasks [1,4,5,8]. Looking more closely at this literature, though, reveals several gaps that are especially acute in the Indian school mathematics context.

First, research on mathematics learning in India has mostly centred on achievement outcomes, procedural proficiency, and examination performance, with comparatively little attention paid to the dispositional side of mathematics learning – students' willingness to attempt unfamiliar problems, or to voice partial reasoning, for instance. The affective side of mathematics learning, particularly how emotional safety relates to academic risk-taking, remains underexplored specifically within the Indian context, even though the underlying constructs – mathematics anxiety, growth mindset, error beliefs – are well studied elsewhere.

Second, NCERT's curricular frameworks call for learner-centred approaches, fear-free classrooms, and using errors as learning opportunities [11,12], but a gap remains between what policy expects and what happens in classrooms, and little classroom-based research has looked at how these principles play out as practical teacher strategies.

Third, international studies provide plenty of theoretical grounding in mathematics anxiety, growth mindset, and productive struggle [5,6,16], but far fewer of them – in India or elsewhere – offer teachers structured pedagogical frameworks for putting risk-tolerant mathematical practices into effect in classrooms shaped by large class sizes and examination pressure.

Finally, only limited research has examined how these patterns vary across different Indian school contexts, including different institutional settings and curricular boards. Understanding how classroom culture, teacher practice, and institutional factors shape students' willingness to take mathematical risks is essential for developing pedagogical approaches sensitive to context, and marks a clear direction for future empirical work.

5.3 Implications for Mathematics Classrooms

The literature surveyed here points toward shifting mathematics classrooms away from performance-oriented environments and toward spaces that encourage exploration, reasoning, and conceptual understanding. Building emotionally safe learning environments takes deliberate pedagogical choices that encourage students to attempt unfamiliar problems without fearing a negative verdict; teachers are central to this through how they respond to student errors, how they question, and how they assess.

One clear implication is adopting error-friendly pedagogy, in which mistakes become an opportunity for discussion and conceptual clarification. Warshauer's finding that teacher scaffolding, rather than removing the challenge, is what makes struggle productive [16] points to a concrete, trainable teacher skill rather than a vague disposition: teachers can be coached in exactly how to respond when a student gets stuck, neither rescuing them nor leaving them to flounder.

Open-ended and non-routine problems are particularly good at encouraging mathematical exploration, especially for conceptually demanding topics like algebra, geometry, trigonometry, and quadratic equations. Teacher facilitation, too, needs to move past quick judgements of correctness and toward process-oriented feedback – giving students time to think, asking probing questions, and acknowledging partial reasoning all build student confidence and participation. Collaborative structures such as peer discussion and group problem-solving add further support to a positive classroom climate.

Reflective prompts, asking students to describe how they tackled a problem, what difficulties came up, and how their understanding changed, can help build self-regulated learning and persistence. Structured teaching modules would help teachers put these practices into place systematically, potentially including facilitation guidelines, reflective prompts, and observation checklists suited to the realities of Indian classrooms. The implications extend to teacher education and professional development as well, where more emphasis should go toward weaving the affective dimensions of mathematics learning into instructional strategy.

6. Limitations of the Review

This review has several limitations. First, because it is a narrative rather than a systematic review, the selection of sources was purposive and thematic rather than exhaustive. It does not offer a full census of the available literature and may miss relevant studies, particularly unpublished work or research not published in English. Second, most of the empirical studies drawn on here were carried out outside India – in

the United States, Europe, and Australia. The theoretical constructs involved, such as mathematics anxiety, growth mindset, and error-friendly pedagogy, are treated as broadly applicable, but none of these studies has directly tested how they operate within Indian classroom cultures, language contexts, or examination systems, so their transfer should not be assumed without qualification. Third, the conceptual framework put forward in this review (Figure 1) synthesises relationships suggested across separate bodies of literature; it has not itself been tested empirically as a whole, and should be read as a hypothesis for future classroom-based or intervention research rather than as an established causal model. Finally, because this is a review of existing literature rather than original empirical work, it cannot establish causal relationships between emotional safety and academic risk-taking – only the consistent associations reported across the studies drawn on here.

7. Conclusion

This review has examined how emotional safety and academic risk-taking relate to one another in mathematics classrooms, with a particular eye on the Indian school context. Bringing together international and national literature suggests that meaningful mathematical learning depends on environments where students feel psychologically secure enough to propose conjectures, attempt unfamiliar problems, voice partial reasoning, and learn from their errors. Risk-taking in mathematics, then, is not simply a trait of the individual student – it is shaped by classroom norms, teacher discourse, assessment practices, and peer interaction [2, 10, 1]. Performance-oriented, examination-driven teaching cultures tend to dampen student participation and reinforce mathematics anxiety. Mastery-oriented environments, by contrast, built on supportive feedback, collaborative problem-solving, and process-based assessment, foster persistence and deeper conceptual understanding [2, 5, 16]. Error-friendly pedagogy sits at the heart of this shift, turning mathematics classrooms into spaces where uncertainty is accepted and exploration is encouraged. Indian curricular frameworks call for fear-free, constructivist mathematics learning, yet this review finds a persisting gap between those policy intentions and classroom practice, matched by a corresponding gap in classroom-based research on Indian settings. Ultimately, promoting academic risk-taking calls for a shift in classroom culture – away from environments where students fear making mistakes, and toward learning communities that treat questioning, experimentation, and reflective thinking as essential to mathematical understanding. The conceptual framework and implications set out here are offered as a starting point for future classroom-based intervention research aimed at closing that gap across diverse Indian school settings.

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